# FS VI: Fuzzy reasoning schemes

$$\begin{array}{ll}
\Re_1: & \text{if } x \text{ is } A_1 \text{ and } y \text{ is } B_1 \text{ then } \quad z \text{ is } C_1 \\
\Re_2: & \text{if } x \text{ is } A_2 \text{ and } y \text{ is } B_2 \text{ then } \quad z \text{ is } C_2 \\
 & \ldots\ldots\ldots\ldots \\
\Re_n: & \text{if } x \text{ is } A_n \text{ and } y \text{ is } B_n \text{ then } \quad z \text{ is } C_n \\
 & \\
 & x \text{ is } \bar{x}_0 \text{ and } y \text{ is } \bar{y}_0 \\
\hline
 & \qquad\qquad\qquad\qquad\qquad\quad z \text{ is } C
\end{array}$$

The $i$-th fuzzy rule from this rule-base

$$\Re_i : \text{if } x \text{ is } A_i \text{ and } y \text{ is } B_i \text{ then } z \text{ is } C_i$$

is implemented by a *fuzzy relation* $R_i$ and is defined as

$$R_i(u, v, w) = (A_i \times B_i \to C_i)(u, w)$$
$$= [A_i(u) \wedge B_i(v)] \to C_i(w)$$

for $i = 1, \ldots, n$.

Find $C$ from the input $x_0$ and from the rule base

$$\Re = \{\Re_1, \dots, \Re_n\}.$$

Interpretation of

- logical connective ”and”
- sentence connective ”also”
- implication operator ”then”
- compositional operator ”$\circ$”

We first compose $\bar{x}_0 \times \bar{y}_0$ with each $R_i$ producing intermediate result

$$C_i' = \bar{x}_0 \times \bar{y}_0 \circ R_i$$

for $i = 1, \dots, n$. Here $C_i'$ is called the output of the $i$-th rule

$$C_i'(w) = [A_i(x_0) \wedge B_i(y_0)] \rightarrow C_i(w),$$

for each $w$.

Then combine the $C'_i$ component wise into $C'$ by some aggregation operator:

$$C = \bigcup_{i=1}^{n} C'_i = \bar{x}_0 \times \bar{y}_0 \circ R_1 \cup \cdots \cup \bar{x}_0 \times \bar{y}_0 \circ R_n$$

$$C(w) = A_1(x_0) \times B_1(y_0) \to C_1(w) \vee \cdots \vee$$
$$A_n(x_0) \times B_n(y_0) \to C_n(w).$$

- input to the system is $(x_0, y_0)$
- fuzzified input is $(\bar{x}_0, \bar{y}_0)$
- firing strength of the $i$-th rule is

$$A_i(x_0) \wedge B_i(y_0)$$

- the $i$-th individual rule output is

$$C'_i(w) := [A_i(x_0) \wedge B_i(y_0)] \to C_i(w)$$

- overall system output is

$$C = C'_1 \cup \cdots \cup C'_n.$$

overall system output = union of the individual rule outputs

We present five well-known inference mechanisms in fuzzy rule-based systems.
For simplicity we assume that we have two fuzzy IF-THEN rules of the form

| | | |
|---|---|---|
| $\Re_1$ : | if $x$ is $A_1$ and $y$ is $B_1$ then | $z$ is $C_1$ |
| also | | |
| $\Re_2$ : | if $x$ is $A_2$ and $y$ is $B_2$ then | $z$ is $C_2$ |
| fact : | $x$ is $\bar{x}_0$ and $y$ is $\bar{y}_0$ | |
| consequence : | | $z$ is $C$ |

**Mamdani.** The fuzzy implication is modelled by Mamdani's minimum operator and the sentence connective *also* is interpreted as oring the propositions and defined by $\max$ operator.
The firing levels of the rules, denoted by $\alpha_i,\ i = 1, 2$, are computed by

$$\alpha_1 = A_1(x_0) \wedge B_1(y_0),$$

$$\alpha_2 = A_2(x_0) \wedge B_2(y_0)$$

The individual rule outputs are obtained by

$$C_1'(w) = (\alpha_1 \wedge C_1(w)),$$

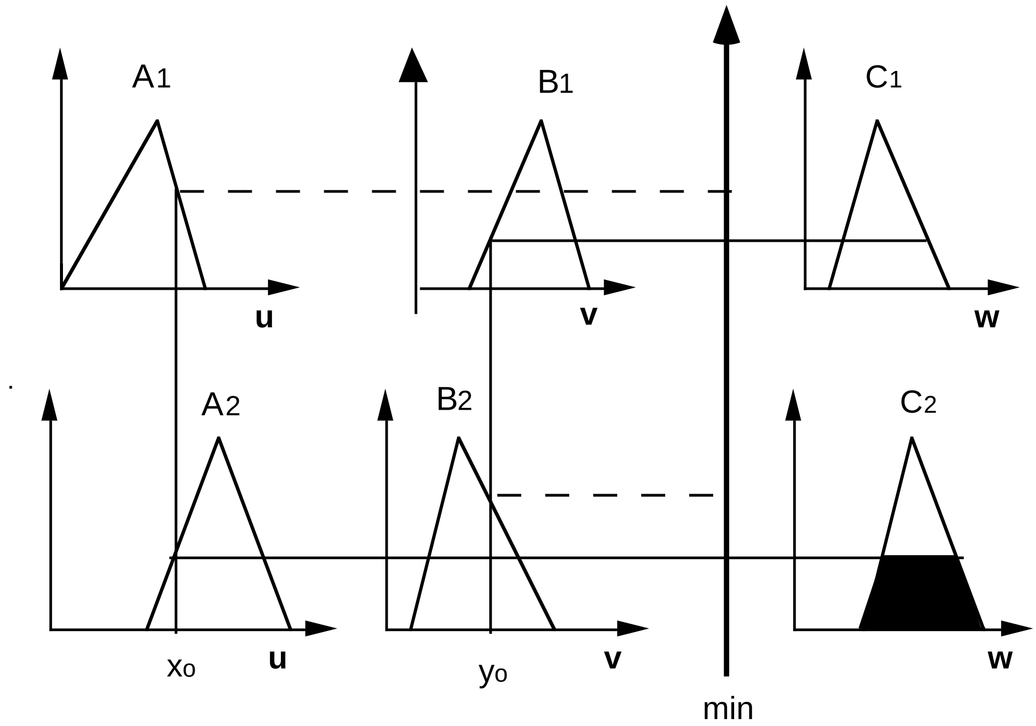
$$C_2'(w) = (\alpha_2 \wedge C_2(w))$$

Then the overall system output is computed by oring the individual rule outputs

$$C(w) = C_1'(w) \vee C_2'(w)$$

$$= (\alpha_1 \wedge C_1(w)) \vee (\alpha_2 \wedge C_2(w))$$

Finally, to obtain a deterministic control action, we employ any defuzzification strategy.




Inference with Mamdani's implication operator.

**Tsukamoto.** All linguistic terms are supposed to have monotonic membership functions.

The firing levels of the rules, denoted by $\alpha_i,\ i = 1, 2$, are computed by

$$\alpha_1 = A_1(x_0) \wedge B_1(y_0),\ \alpha_2 = A_2(x_0) \wedge B_2(y_0)$$

In this mode of reasoning the individual crisp control actions $z_1$ and $z_2$ are computed from the equations

$$\alpha_1 = C_1(z_1), \quad \alpha_2 = C_2(z_2)$$

and the overall crisp control action is expressed as

$$z_0 = \frac{\alpha_1 z_1 + \alpha_2 z_2}{\alpha_1 + \alpha_2}$$

i.e. $z_0$ is computed by the discrete Center-of-Gravity method.
If we have $n$ rules in our rule-base then the crisp control action is computed as

$$z_0 = \frac{\sum_{i=1}^{n} \alpha_i z_i}{\sum_{i=1}^{n} \alpha_i},$$

where $\alpha_i$ is the firing level and $z_i$ is the (crisp) output of the $i$-th rule, $i = 1, \dots, n$

**Example 1.** *We illustrate Tsukamoto's reasoning method by the following simple example*

| | | |
|---|---|---|
| $\Re_1$ : | *if* $x$ *is* $A_1$ *and* $y$ *is* $B_1$ *then* | $z$ *is* $C_1$ |
| also | | |
| $\Re_2$ : | *if* $x$ *is* $A_2$ *and* $y$ *is* $B_2$ *then* | $z$ *is* $C_2$ |
| *fact* : | $x$ *is* $\bar{x}_0$ *and* $y$ *is* $\bar{y}_0$ | |
| *consequence* : | | $z$ *is* $C$ |

Then according to the figure we see that

$$A_1(x_0) = 0.7, \quad B_1(y_0) = 0.3$$

therefore, the firing level of the first rule is

$$\alpha_1 = \min\{A_1(x_0), B_1(y_0)\}$$
$$= \min\{0.7, 0.3\} = 0.3$$

and from

$$A_2(x_0) = 0.6, \quad B_2(y_0) = 0.8$$

it follows that the firing level of the second rule is

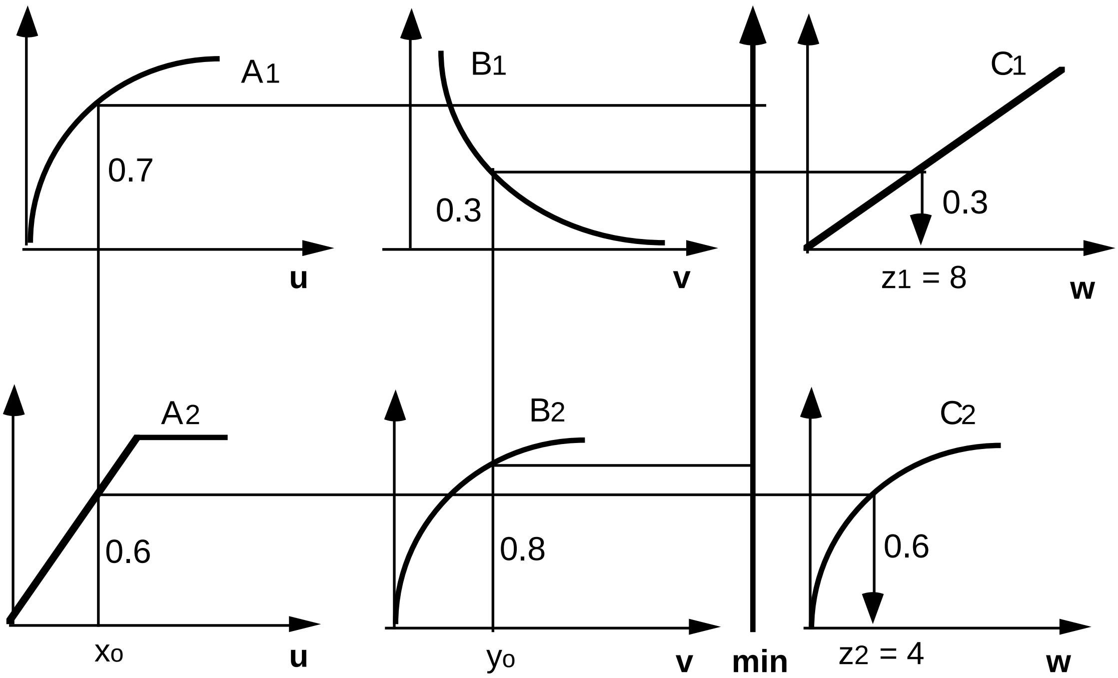
$$\alpha_2 = \min\{A_2(x_0), B_2(y_0)\}$$
$$= \min\{0.6, 0.8\} = 0.6,$$

the individual rule outputs $z_1 = 8$ and $z_2 = 4$ are derived from the equations

$$C_1(z_1) = 0.3, \quad C_2(z_2) = 0.6$$

and the crisp control action is

$$z_0 = (8 \times 0.3 + 4 \times 0.6)/(0.3 + 0.6) = 6.$$




Tsukamoto's inference mechanism.

**Sugeno.** Sugeno and Takagi use the following architecture

| | | |
|---|---|---|
| $\Re_1$ : | if $x$ is $A_1$ and $y$ is $B_1$ then | $z_1 = a_1x + b_1y$ |
| also | | |
| $\Re_2$ : | if $x$ is $A_2$ and $y$ is $B_2$ then | $z_2 = a_2x + b_2y$ |
| fact : | $x$ is $\bar{x}_0$ and $y$ is $\bar{y}_0$ | |
| cons. : | | $z_0$ |

The firing levels of the rules are computed by

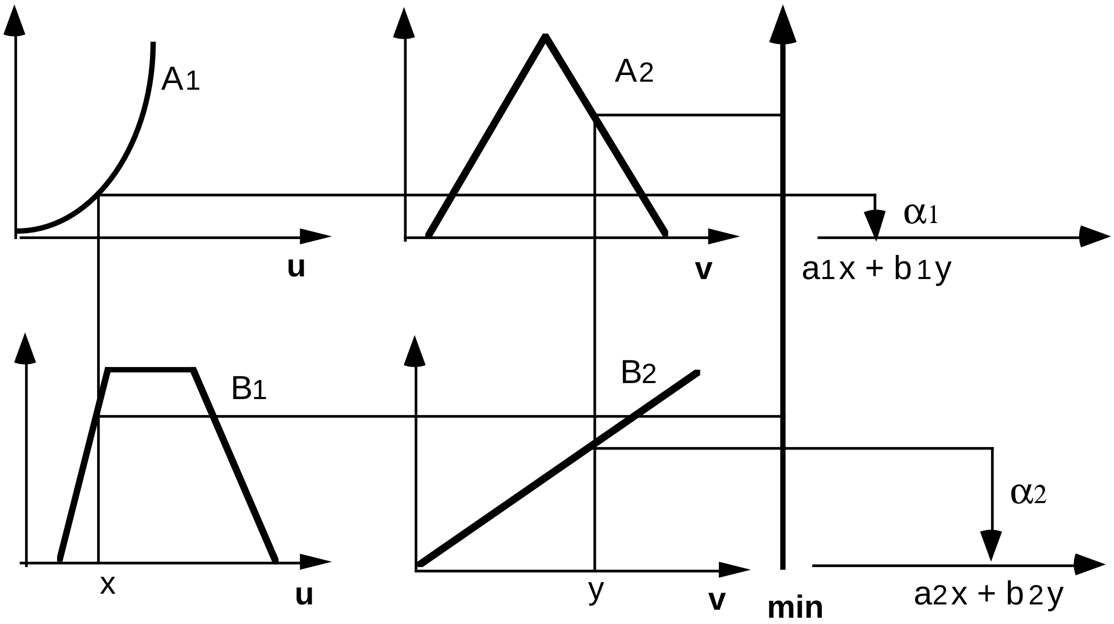
$$\alpha_1 = A_1(x_0) \wedge B_1(y_0),\ \alpha_2 = A_2(x_0) \wedge B_2(y_0)$$

then the individual rule outputs are derived from the relationships

$$z_1^* = a_1x_0 + b_1y_0,\ z_2^* = a_2x_0 + b_2y_0$$

and the crisp control action is expressed as

$$z_0 = \frac{\alpha_1 z_1^* + \alpha_2 z_2^*}{\alpha_1 + \alpha_2}$$




Sugeno's inference mechanism.

If we have $n$ rules in our rule-base then the crisp control action is computed as

$$z_0 = \frac{\sum_{i=1}^{n} \alpha_i z_i^*}{\sum_{i=1}^{n} \alpha_i},$$

where $\alpha_i$ denotes the firing level of the $i$-th rule, $i = 1, \dots, n$

**Example 2.** *We illustrate Sugeno's reasoning method by the following simple example*

| | | |
|---|---|---|
| $\Re_1$ : | *if* $x$ *is BIG and* $y$ *is SMALL then* | $z_1 = x + y$ |
| also | | |
| $\Re_2$ : | *if* $x$ *is MEDIUM and* $y$ *is BIG then* | $z_2 = 2x - y$ |
| *fact* : | $x_0$ *is 3 and* $y_0$ *is 2* | |
| *conseq* : | | $z_0$ |

Then according to the figure we see that

$$\mu_{BIG}(x_0) = \mu_{BIG}(3) = 0.8,$$

$$\mu_{SMALL}(y_0) = \mu_{SMALL}(2) = 0.2$$

therefore, the firing level of the first rule is

$$\alpha_1 = \min\{\mu_{BIG}(x_0), \mu_{SMALL}(y_0)\}$$

$$= \min\{0.8, 0.2\} = 0.2$$

and from

$$\mu_{MEDIUM}(x_0) = \mu_{MEDIUM}(3) = 0.6,$$

$$\mu_{BIG}(y_0) = \mu_{BIG}(2) = 0.9$$

it follows that the firing level of the second rule is

$$\alpha_2 = \min\{\mu_{MEDIUM}(x_0), \mu_{BIG}(y_0)\}$$

$$= \min\{0.6, 0.9\} = 0.6.$$

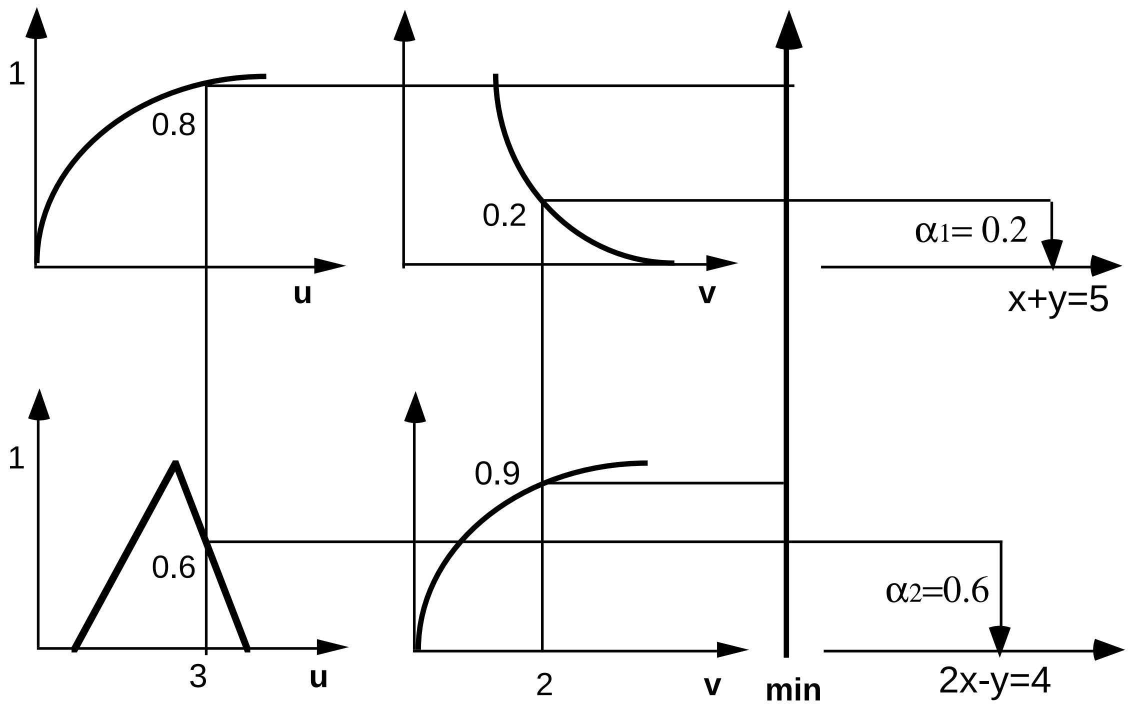
the individual rule outputs are computed as

$$z_1^* = x_0 + y_0 = 3 + 2 = 5, \; z_2^*$$

$$= 2x_0 - y_0 = 2 \times 3 - 2 = 4$$

so the crisp control action is

$$z_0 = (5 \times 0.2 + 4 \times 0.6)/(0.2 + 0.6) = 4.25.$$




Example of Sugeno's inference mechanism.

**Larsen.** The fuzzy implication is modelled

by Larsen's prduct operator and the sentence connective *also* is interpreted as oring the propositions and defined by $\max$ operator.
Let us denote $\alpha_i$ the firing level of the $i$-th rule, $i = 1, 2$

$$\alpha_1 = A_1(x_0) \wedge B_1(y_0),$$

$$\alpha_2 = A_2(x_0) \wedge B_2(y_0).$$

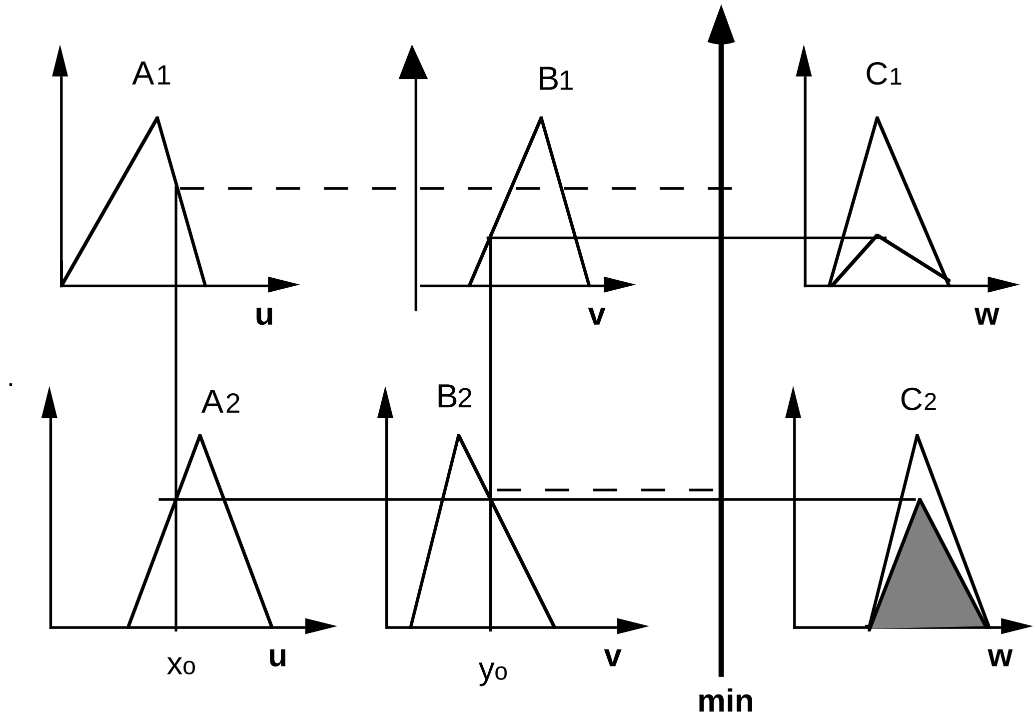
Then membership function of the inferred consequence $C$ is pointwise given by

$$C(w) = (\alpha_1 C_1(w)) \vee (\alpha_2 C_2(w)).$$

To obtain a deterministic control action, we employ any defuzzification strategy.
If we have $n$ rules in our rule-base then the consequence $C$ is computed as

$$C(w) = \bigvee_{i=1}^{n} (\alpha_i C_1(w))$$

where $\alpha_i$ denotes the firing level of the $i$-th rule, $i = 1, \dots, n$




Inference with Larsen's product operation rule.

**Simplified fuzzy reasoning**

| | |
|---|---|
| $\Re_1$ : | if $x$ is $A_1$ and $y$ is $B_1$ then $z_1 = c_1$ |
| also | |
| $\Re_2$ : | if $x$ is $A_2$ and $y$ is $B_2$ then $z_2 = c_2$ |
| fact : | $x$ is $\bar{x}_0$ and $y$ is $\bar{y}_0$ |
| consequence : | $z_0$ |

The firing levels of the rules are computed by

$$\alpha_1 = A_1(x_0) \wedge B_1(y_0),$$

$$\alpha_2 = A_2(x_0) \wedge B_2(y_0)$$

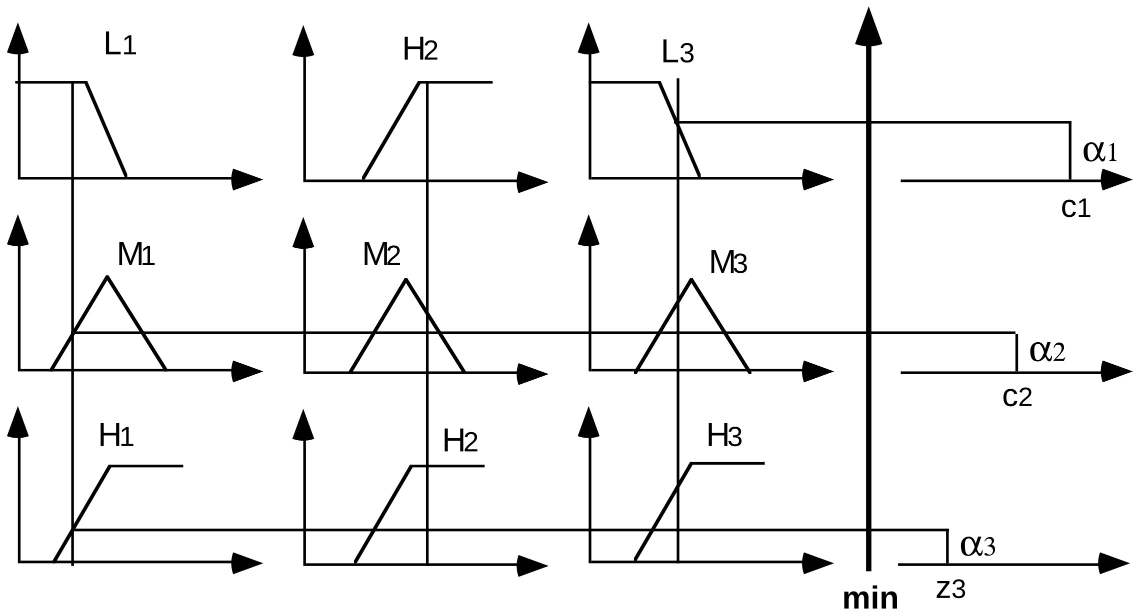
then the individual rule outputs are $c_1$ and $c_2$, and the crisp control action is expressed as

$$z_0 = \frac{\alpha_1 c_1 + \alpha_2 c_2}{\alpha_1 + \alpha_2}$$

If we have $n$ rules in our rule-base then the crisp control action is computed as

$$z_0 = \frac{\sum_{i=1}^n \alpha_i c_i}{\sum_{i=1}^n \alpha_i},$$

where $\alpha_i$ denotes the firing level of the $i$-th rule, $i = 1, \dots, n$


L1
H2
L3
α1
c1
M1
M2
M3
α2
c2
H1
H2
H3
α3
min
z3