FS IV: The theory of approximate reasoning

In 1979 *Zadeh* introduced the theory of approximate reasoning. This theory provides a powerful framework for reasoning in the face of imprecise and uncertain information.

Central to this theory is the representation of propositions as statements assigning fuzzy sets as values to variables.

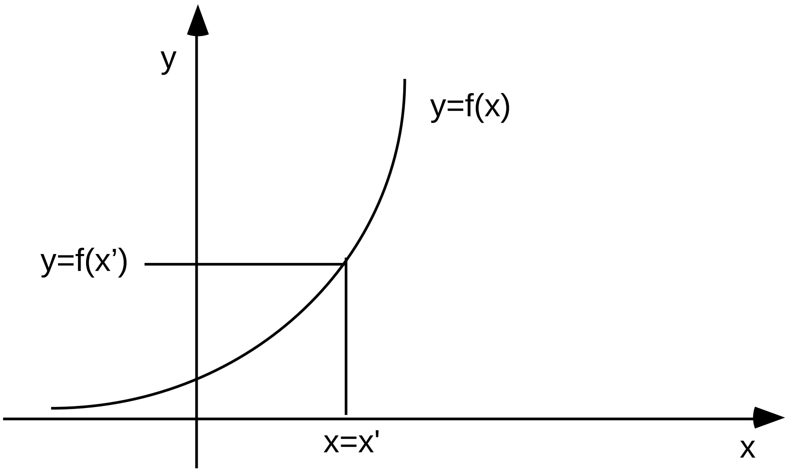
Suppose we have two interactive variables $x \in X$ and $y \in Y$ and the causal relationship between x and y is completely known. Namely, we know that y is a function of x

$$y = f(x)$$

Then we can make inferences easily

premise	$y = f(x)$
fact	$x = x'$
consequence	$y = f(x')$

This inference rule says that if we have $y = f(x), \forall x \in X$ and we observe that $x = x'$ then y takes the value $f(x')$.



Simple crisp inference.

More often than not we do not know the complete causal link f between x and y, only we now the values of $f(x)$ for some particular values of x

$$\Re_1: \quad \text{If } x = x_1 \text{ then } y = y_1$$

also

$$\Re_2: \quad \text{If } x = x_2 \text{ then } y = y_2$$

also

$$\dots$$

also

$$\Re_n: \quad \text{If } x = x_n \text{ then } y = y_n$$

Suppose that we are given an $x' \in X$ and want to find an $y' \in Y$ which correponds to x' under the rule-base.

$\Re_1$:	If	$x = x_1$ then	$y = y_1$
also			
$\Re_2$:	If	$x = x_2$ then	$y = y_2$
also			
	$\dots$		$\dots$
also			
$\Re_n$:	If	$x = x_n$ then	$y = y_n$
fact:		$x = x'$	
consequence:			$y = y'$

This problem is frequently quoted as interpolation.

Let x and y be linguistic variables, e.g. "x is high" and "y is small".

The basic problem of approximate reasoning is to find the membership function of the consequence C

from the rule-base $\{\Re_1, \ldots, \Re_n\}$ and the fact A.

$$
\begin{array}{ll}
\Re_1 : & \text{if } x \text{ is } A_1 \text{ then } y \text{ is } C_1, \\
\Re_2 : & \text{if } x \text{ is } A_2 \text{ then } y \text{ is } C_2, \\
 & \ldots\ldots\ldots\ldots \\
\Re_n : & \text{if } x \text{ is } A_n \text{ then } y \text{ is } C_n \\
\text{fact:} & \quad x \text{ is } A \\
\hline
\text{consequence:} & \qquad\qquad y \text{ is } C
\end{array}
$$

Zadeh introduced a number of translation rules which allow us to represent some common linguistic statements in terms of propositions in our language.

In the following we describe some of these translation rules.

Definition 1. *Entailment rule:*

$$
\begin{array}{lcl}
x \textit{ is } A & \qquad & \textit{Mary is very young} \\
A \subset B & & \textit{very young} \subset \textit{young} \\
\hline
x \textit{ is } B & & \textit{Mary is young}
\end{array}
$$

Definition 2. *Conjuction rule:*

$$\frac{\begin{array}{l} x \textit{ is } A \\ x \textit{ is } B \end{array}}{x \textit{ is } A \cap B}$$

$$\frac{\begin{array}{l} \textit{pressure is not very high} \\ \textit{pressure is not very low} \end{array}}{\textit{pressure is not very high and not very low}}$$

Definition 3. *Disjunction rule:*

$$\frac{\begin{array}{ll} & x \textit{ is } A \\ \textit{or} & x \textit{ is } B \end{array}}{\quad x \textit{ is } A \cup B}$$

$$\frac{\begin{array}{ll} & \textit{pressure is not very high vspace4pt} \\ \textit{or} & \textit{pressure is not very low} \end{array}}{\quad \textit{pressure is not very high or not very low}}$$

Definition 4. *Projection rule:*

$$\frac{(x, y) \text{ have relation } R}{x \text{ is } \Pi_X(R)}$$

$$\frac{(x, y) \text{ have relation } R}{y \text{ is } \Pi_Y(R)}$$

$$\frac{(x, y) \text{ is close to } (3, 2)}{x \text{ is close to } 3} \qquad \frac{(x, y) \text{ is close to } (3, 2)}{y \text{ is close to } 2}$$

Definition 5. *Negation rule:*

$$\frac{\text{not } (x \text{ is } A)}{x \text{ is } \neg A} \qquad \frac{\text{not } (x \text{ is high})}{x \text{ is not high}}$$

In fuzzy logic and approximate reasoning, the most important fuzzy implication inference rule is the *Generalized Modus Ponens* (GMP). The classical *Modus Ponens* inference rule says:

premise	if p then	q
fact	p	
consequence		q

This inference rule can be interpreted as: If p is true and $p \to q$ is true then q is true.

The fuzzy implication inference is based on the compositional rule of inference for approximate reasoning suggested by Zadeh.

Definition 6. *(compositional rule of inference)*

premise	*if* x *is* A *then*	y *is* B
fact	x *is* A'	
consequence:		y *is* B'

where the consequence B' *is determined as a composition of the fact and the fuzzy implication opera-*

tor

$$B' = A' \circ (A \to B)$$

that is,

$$B'(v) = \sup_{u \in U} \min\{A'(u), (A \to B)(u, v)\}, \; v \in V.$$

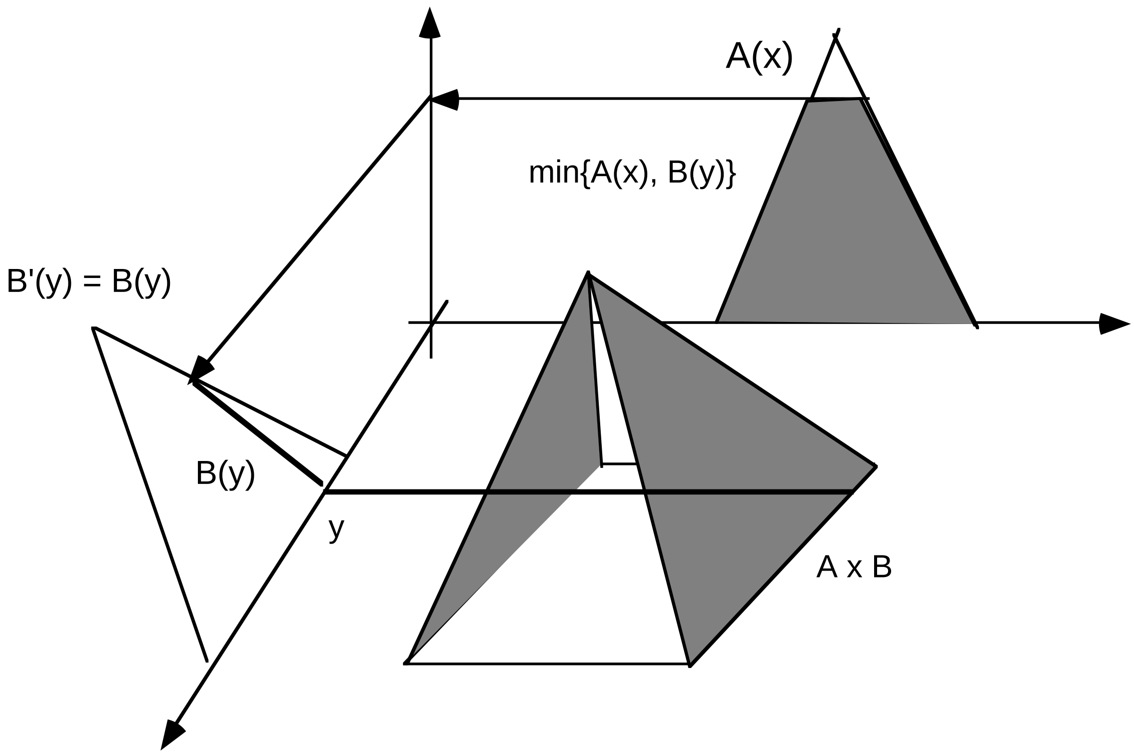
The consequence B' is nothing else but the shadow of $A \to B$ on A'.

The *Generalized Modus Ponens*, which reduces to calssical modus ponens when $A' = A$ and $B' = B$, is closely related to the forward data-driven inference which is particularly useful in the *Fuzzy Logic Control*.

The classical *Modus Tollens inference rule* says: If $p \to q$ is true and q is false then p is false. The *Generalized Modus Tollens*,

premise	if x is A then	y is B
fact		y is B'
consequence:	x is A'	

which reduces to ”Modus Tollens” when $B = \neg B$ and $A' = \neg A$, is closely related to the backward goal-driven inference which is commonly used in expert systems, especially in the realm of *medical diagnosis*.



$$A \circ A \times B = B.$$

Suppose that A, B and A' are fuzzy numbers. The Generalized Modus Ponens should satisfy some rational properties

Property 1. *Basic property:*

$$\begin{array}{l} \textit{if}\;\; x \text{ is } A \textit{ then }\; y \text{ is } B \\ \quad\;\; x \text{ is } A \\ \hline \qquad\qquad\qquad\qquad y \text{ is } B \end{array}$$

if pressure is big *then* volume is small
pressure is big

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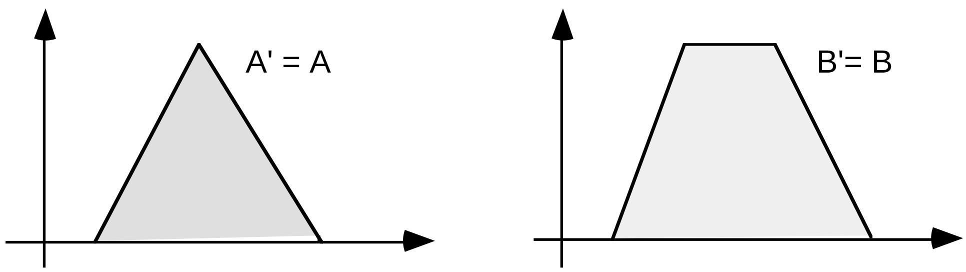
volume is small

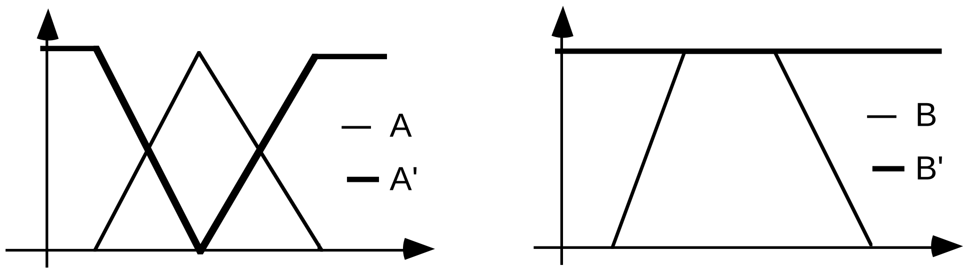


Basic property.

Property 2. *Total indeterminance:*

$$\frac{\begin{array}{l} \textit{if } x \textit{ is } A \textit{ then } \; y \textit{ is } B \\ x \textit{ is } \neg A \end{array}}{\hfill y \textit{ is } unknown}$$

if pres. is big	*then* volume is small	
pres. is not big		
	volume is unknown	

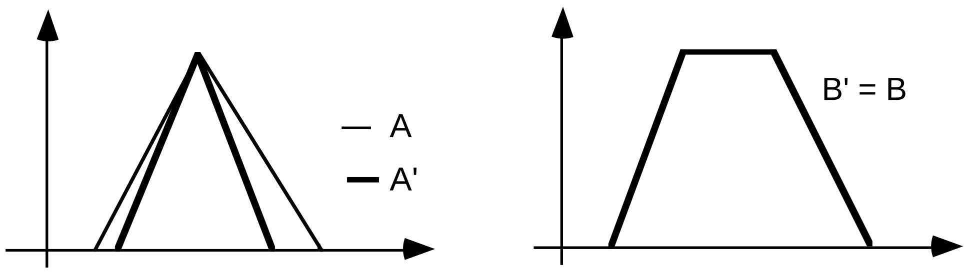


Total indeterminance.

Property 3. *Subset:*

$$\begin{array}{l}
\textit{if } x \textit{ is } A \textit{ then } \; y \textit{ is } B \\
x \textit{ is } A' \subset A \\
\hline
\qquad\qquad\qquad\qquad y \textit{ is } B
\end{array}$$

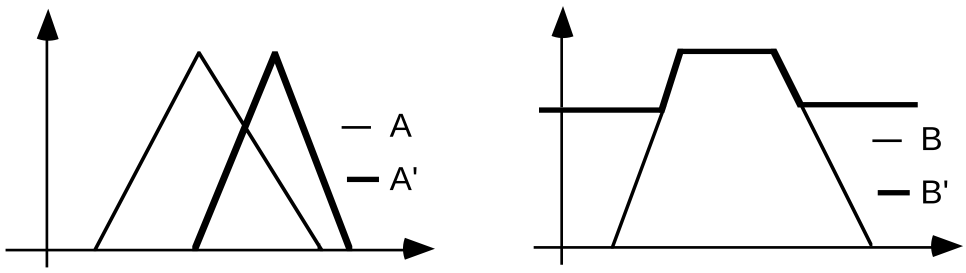
$$\begin{array}{ll}
\textit{if} & \text{pressure is big } \textit{then} \;\; \text{volume is small} \\
 & \text{pressure is very big} \\
\hline
 & \qquad\qquad\qquad\qquad \text{volume is small}
\end{array}$$



Subset property.

Property 4. *Superset:*

$$\begin{array}{l} \textit{if} \;\; x \textit{ is } A \textit{ then } \; y \textit{ is } B \\ \quad\;\; x \textit{ is } A' \\ \hline \qquad\qquad\qquad y \textit{ is } B' \supset B \end{array}$$



Superset property.

Suppose that A, B and A' are fuzzy numbers.

We show that the Generalized Modus Ponens with Mamdani's implication operator does not satisfy all the four properties listed above.

Example 1. *(The GMP with Mamdani implication)*

$$
\begin{array}{l}
\textit{if } \; x \textit{ is } A \textit{ then } \; y \textit{ is } B \\
\quad\; x \textit{ is } A' \\
\hline
\qquad\qquad\qquad\quad\; y \textit{ is } B'
\end{array}
$$

where the membership function of the consequence B' *is defined by*

$$B'(y) = \sup\{A'(x) \wedge A(x) \wedge B(y) | x \in \mathbf{R}\}, \; y \in \mathbf{R}.$$

Basic property. Let $A' = A$ and let $y \in \mathbf{R}$ be arbitrarily fixed. Then we have

$$B'(y) = \sup_x \min\{A(x), \min\{A(x), B(y)\} =$$

$$\sup_x \min\{A(x), B(y)\} = \min\{B(y), \sup_x A(x)\} =$$

$$\min\{B(y), 1\} = B(y).$$

So the basic property is satisfied.

Total indeterminance. Let $A' = \neg A = 1 - A$ and let $y \in \mathbf{R}$ be arbitrarily fixed. Then we have

$$B'(y) = \sup_x \min\{1 - A(x), \min\{A(x), B(y)\} =$$

$$\sup_x \min\{A(x), 1 - A(x), B(y)\} =$$

$$\min\{B(y), \sup_x \min\{A(x), 1 - A(x)\}\} =$$

$$\min\{B(y), 1/2\} = 1/2B(y) < 1$$

this means that the total indeterminance property is not satisfied.

Subset. Let $A' \subset A$ and let $y \in \mathbf{R}$ be arbitrarily fixed. Then we have

$$B'(y) = \sup_x \min\{A'(x), \min\{A(x), B(y)\} =$$

$$\sup_x \min\{A(x), A'(x), B(y)\} =$$

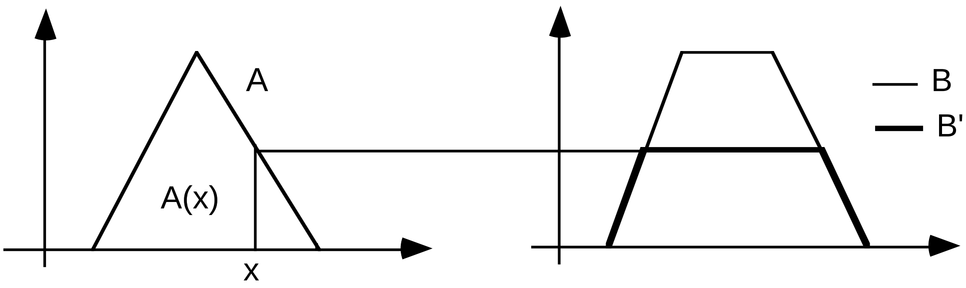
$$\min\{B(y), \sup_x A'(x)\} = \min\{B(y), 1\} = B(y)$$

So the subset property is satisfied.

Superset. Let $y \in \mathbf{R}$ be arbitrarily fixed. Then we have

$$B'(y) = \sup_x \min\{A'(x), \min\{A(x), B(y)\} =$$
$$\sup_x \min\{A(x), A'(x), B(y)\} \leq B(y).$$

So the superset property of GMP is not satisfied by Mamdani's implication operator.



The GMP with Mamdani's implication operator.

Example 2. *(The GMP with Larsen's product implication)*

$$\begin{array}{ll} \textit{if} & x \textit{ is } A \textit{ then } \; y \textit{ is } B \\ & x \textit{ is } A' \\ \hline & \qquad\qquad y \textit{ is } B' \end{array}$$

where the membership function of the consequence B' *is defined by*

$$B'(y) = \sup\min\{A'(x), A(x)B(y) | x \in \mathbf{R}\},\ y \in \mathbf{R}.$$

Basic property. Let $A' = A$ and let $y \in \mathbf{R}$ be arbitrarily fixed. Then we have

$$B'(y) = \sup_x \min\{A(x), A(x)B(y)\} = B(y).$$

So the basic property is satisfied.

Total indeterminance. Let $A' = \neg A = 1 - A$ and let $y \in \mathbf{R}$ be arbitrarily fixed. Then we have

$$B'(y) = \sup_x \min\{1 - A(x), A(x)B(y)\}$$

$$= \frac{B(y)}{1 + B(y)} < 1$$

this means that the total indeterminance property is not satisfied.

Subset. Let $A' \subset A$ and let $y \in \mathbf{R}$ be arbitrarily fixed. Then we have

$$B'(y) = \sup_x \min\{A'(x), A(x)B(y)\} =$$

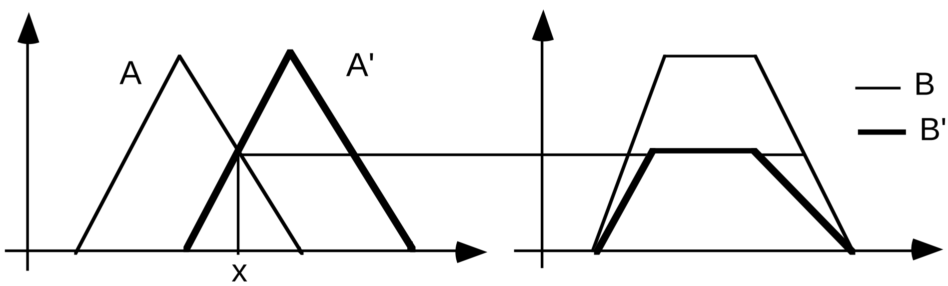
$$\sup_x \min\{A(x), A'(x)B(y)\} = B(y)$$

So the subset property is satisfied.

Superset. Let $y \in \mathbf{R}$ be arbitrarily fixed. Then we have

$$B'(y) = \sup_x \min\{A'(x), A(x)B(y)\} \le B(y).$$

So, the superset property is not satisfied.



The GMP with Larsen's implication operator.