FS III: Fuzzy implications

The material implication

Let $p = {}$"x is in A" and $q = {}$"y is in B" are crisp propositions, where A and B are crisp sets for the moment.

The implication $p \to q$ is interpreted as $\neg(p \wedge \neg q)$.

"p entails q" means that it can never happen that p is true and q is not true.

It is easy to see that

$$p \to q = \neg p \vee q$$

The full interpretation of the material implication $p \to q$ is that the degree of truth of $p \to q$ quantifies to what extend q is at least as true as p, i.e.

$$p \to q \text{ is true} \iff \tau(p) \leq \tau(q)$$

$$p \rightarrow q = \begin{cases} 1 & \text{if } \tau(p) \leq \tau(q) \\ 0 & \text{otherwise} \end{cases}$$

p	q	$p \rightarrow q$
1	1	1
0	1	1
0	0	1
1	0	0

The truth table for the material implication.

Example 1. *Let* $p =$ "x *is bigger than* 10" *and let* $q =$ "x *is bigger than* 9". *It is easy to see that* $p \rightarrow q$ *is true, because it can never happen that* x *is bigger than 10 and* x *is not bigger than 9.*

This property of material implication can be interpreted as:

$$\text{if } X \subset Y \text{ then } X \rightarrow Y$$

Other interpretation of the implication operator is

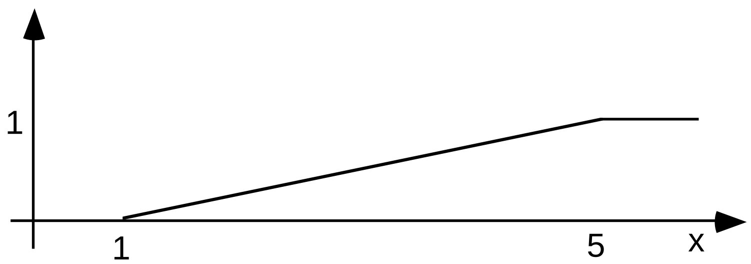
$$X \to Y = \sup\{Z | X \cap Z \subset Y\}.$$

Fuzzy implications

Consider the implication statement

if pressure is high then volume is small

The membership function of the fuzzy set A, *big pressure*, illustrated in the figure



Membership function for ”big pressure”.

can be interpreted as

- 1 is in the fuzzy set *big pressure* with grade of membership 0
- 2 is in the fuzzy set *big pressure* with grade of membership 0.25
- 4 is in the fuzzy set *big pressure* with grade of membership 0.75
- x is in the fuzzy set *big pressure* with grade of membership 1, for all $x \geq 5$

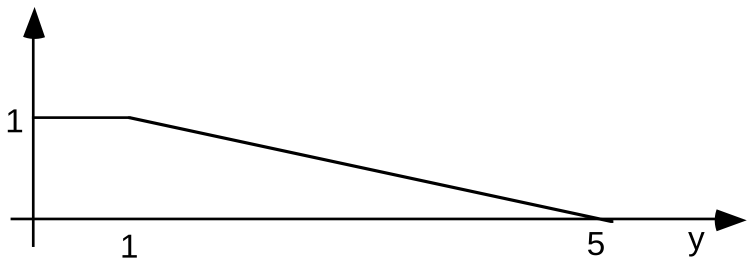
$$A(u) = \begin{cases} 1 & \text{if } u \geq 5 \\ 1 - \dfrac{5-u}{4} & \text{if } 1 \leq u \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

The membership function of the fuzzy set B, *small volume*, can be interpreted as (see figure)

- 5 is in the fuzzy set *small volume* with grade of membership 0

- 4 is in the fuzzy set *small volume* with grade of membership 0.25
- 2 is in the fuzzy set *small volume* with grade of membership 0.75
- x is in the fuzzy set *small volume* with grade of membership 1, for all $x \leq 1$

$$B(v) = \begin{cases} 1 & \text{if } v \leq 1 \\ 1 - \dfrac{v-1}{4} & \text{if } 1 \leq v \leq 5 \\ 0 & \text{otherwise} \end{cases}$$



Membership function for ”small volume”.

If p is a proposition of the form

$$x \text{ is } A$$

where A is a fuzzy set, for example, *big pressure* and q is a proposition of the form

$$y \text{ is } B$$

for example, *small volume* then we define the fuzzy implication $A \to B$ as a fuzzy relation.

It is clear that $(A \to B)(u, v)$ should be defined *pointwise* and *likewise*, i.e. $(A \to B)(u, v)$ depends only on $A(u)$ and $B(v)$.

That is

$$(A \to B)(u, v) = I(A(u), B(v)) = A(u) \to B(v)$$

In our interpretation $A(u)$ is considered as the truth value of the proposition

”u is big pressure”,

and $B(v)$ is considered as the truth value of the proposition

$$"v \text{ is small volume.}$$

that is

u is big pressure $\rightarrow v$ is small volume $\equiv A(u) \rightarrow B(v)$

Remembering the full interpretation of the material implication

$$p \rightarrow q = \begin{cases} 1 & \text{if } \tau(p) \leq \tau(q) \\ 0 & \text{otherwise} \end{cases}$$

One possible extension of materioal implication to implications with intermediate truth values can be

$$A(u) \to B(v) = \begin{cases} 1 & \text{if } A(u) \leq B(v) \\ 0 & \text{otherwise} \end{cases}$$

$$\text{"4 is big pressure"} \to \text{"1 is small volume"} =$$
$$A(4) \to B(1) = 0.75 \to 1 = 1$$

However, it is easy to see that this fuzzy implication operator (called Standard Strict) sometimes is not appropriate for real-life applications. Namely, let $A(u) = 0.8$ and $B(v) = 0.8$. Then we have

$$A(u) \to B(v) = 0.8 \to 0.8 = 1$$

Suppose there is a small error of measurement in $B(v)$, and instead of 0.8 we have 0.7999. Then

$$A(u) \to B(v) = 0.8 \to 0.7999 = 0$$

This example shows that small changes in the input can cause a big deviation in the output, i.e. our system is very sensitive to rounding errors of digital computation and small errors of measurement.

A smoother extension of material implication operator can be derived from the equation

$$X \to Y = \sup\{Z | X \cap Z \subset Y\}$$

That is

$$A(u) \to B(v) = \sup\{z | \min\{A(u), z\} \leq B(v)\}$$

so,

$$A(u) \to B(v) = \begin{cases} 1 & \text{if } A(u) \leq B(v) \\ B(v) & \text{otherwise} \end{cases}$$

This operator is called Gödel implication. Other possibility is to extend the original definition,

$$p \to q = \neg p \vee q$$

using the definition of negation and union

$$A(u) \to B(v) = \max\{1 - A(u), B(v)\}$$

This operator is called Kleene-Dienes implication.

In many practical applications they use Mamdani's implication operator to model causal relationship between fuzzy variables.

This operator simply takes the minimum of truth values of fuzzy predicates

$$A(u) \to B(v) = \min\{A(u), B(v)\}$$

It is easy to see this is not a correct extension of material implications, because $0 \to 0$ yields zero. However, in knowledge-based systems, we are usually not interested in rules, where the antecedent part is false.

Larsen	$x \to y = xy$
Łukasiewicz	$x \to y = \min\{1, 1 - x + y\}$
Mamdani	$x \to y = \min\{x, y\}$
Standard Strict	$x \to y = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$
Gödel	$x \to y = \begin{cases} 1 & \text{if } x \leq y \\ \text{y} & \text{otherwise} \end{cases}$
Gaines	$x \to y = \begin{cases} 1 & \text{if } x \leq y \\ \text{y/x} & \text{otherwise} \end{cases}$
Kleene-Dienes	$x \to y = \max\{1 - x, y\}$
Kleene-Dienes-Łuk.	$x \to y = 1 - x + xy$

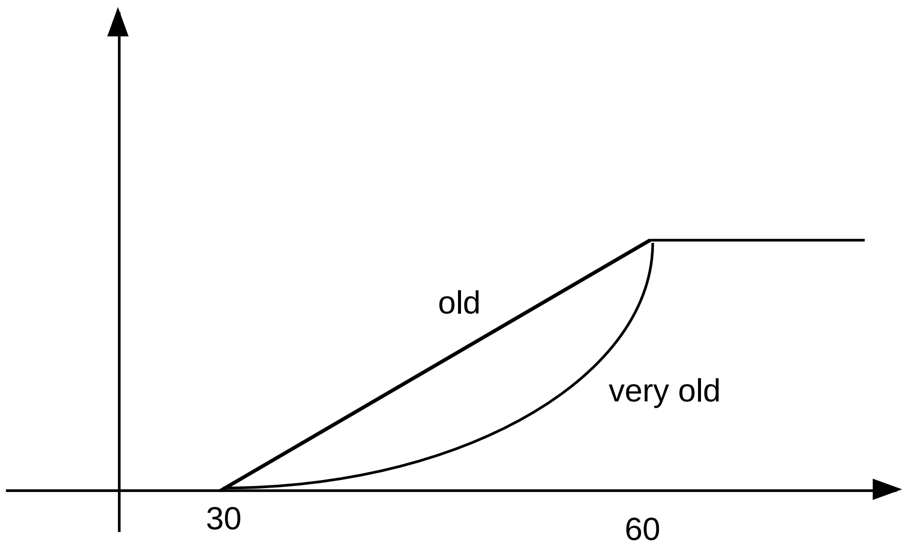
Modifiers

Definition 1. *Let A be a fuzzy set in X. Then we can define the fuzzy sets "very A" and "more or less A" by*

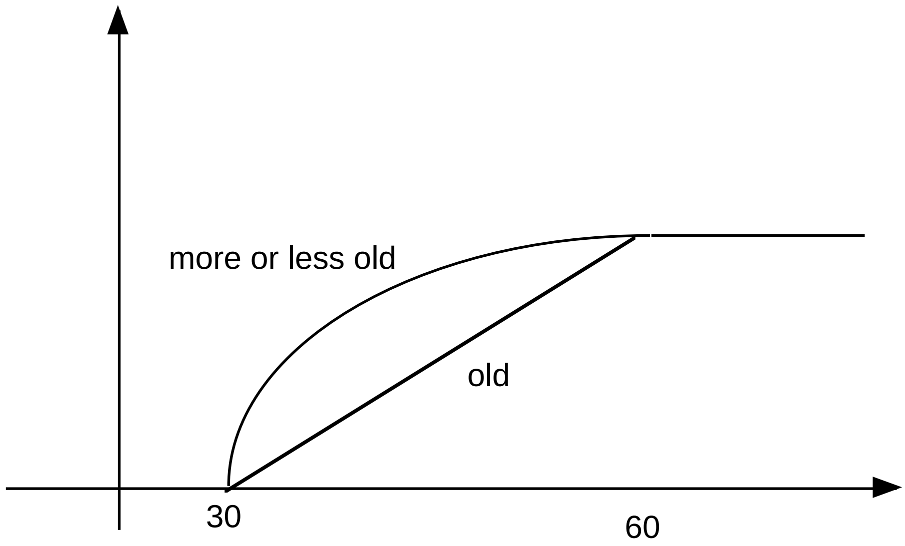
(very A)$(x) = A(x)^2$, *(more or less A)*$(x) = \sqrt{A(x)}$

The use of fuzzy sets provides a basis for a systematic way for the manipulation of vague and imprecise concepts.

In particular, we can employ fuzzy sets to represent linguistic variables.



A linguistic variable can be regarded either as a variable whose value is a fuzzy number or as a variable whose values are defined in linguistic terms.



Definition 2. *Linguistic Variables:*

A linguistic variable is characterized by a quintuple

$$(x, T(x), U, G, M)$$

in which

- x is the name of variable;
- $T(x)$ is the term set of x, that is, the set of names of linguistic values of x with each value being a fuzzy number defined on U;

- G is a syntactic rule for generating the names of values of x;
- and M is a semantic rule for associating with each value its meaning.

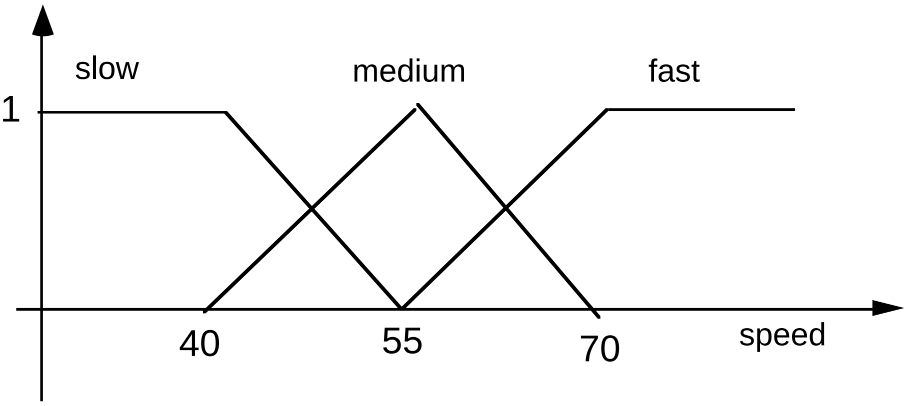
For example, if *speed* is interpreted as a linguistic variable, then its term set T (speed) could be

T = $\{$slow, moderate, fast, very slow, more or less fast, $\ldots\}$

where each term in T (speed) is characterized by a fuzzy set in a universe of discourse $U = [0, 100]$.

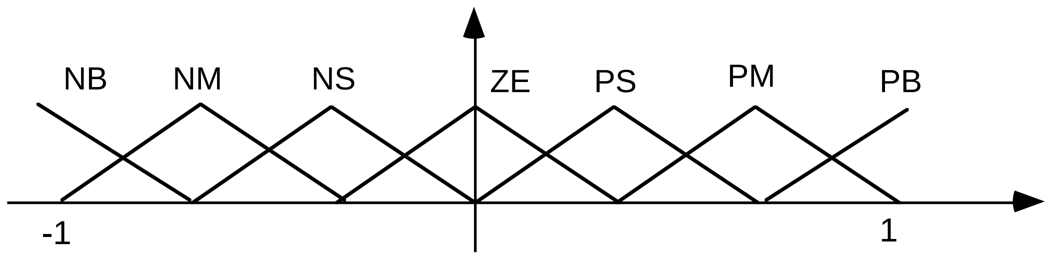
We might interpret ”slow” as ”a speed below about 40 mph,” ”moderate” as ”a speed close to 55 mph,” and ”fast” as ”a speed above about 70mph”.

These terms can be characterized as fuzzy sets whose membership functions are shown in the figure below.



Values of linguistic variable *speed*.

In many practical applications we normalize the domain of inputs and use the following type of fuzzy partition



A possible fuzzy partition of [-1, 1].

Here we used the abbreviations

NB Negative Big, [NM] Negative Medium

NS Negative Small, [ZE] Zero,

PS Positive Small, [PM] Positive Medium,

PB Positive Big.

The linguistic variable Truth.

Truth = $\{$Absolutely false, Very false, False, Fairly true, True, Very true, Absolutely true $\}$

One may define the membership function of linguistic terms of truth as

$$True(u) = u$$

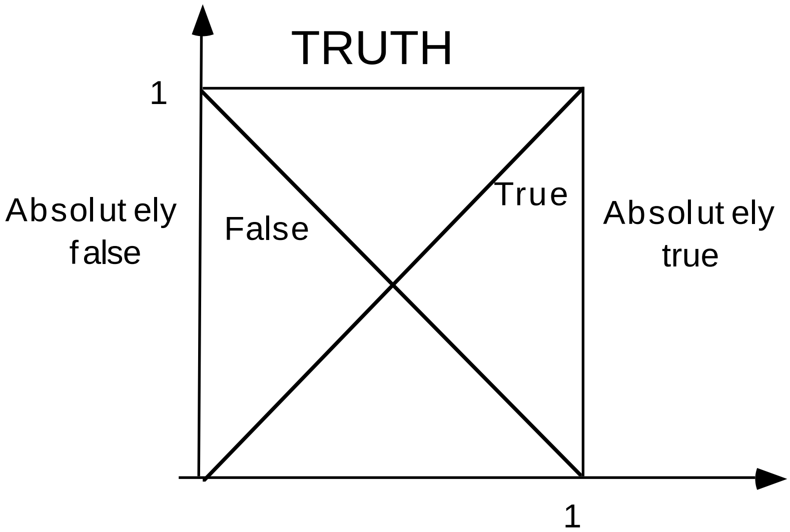
for each $u \in [0, 1]$.

$$False(u) = 1 - u$$

for each $u \in [0,1]$.

$$Absolutely\ false(u) = \begin{cases} 1 & \text{if } u = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$Absolutely\ true(u) = \begin{cases} 1 & \text{if } u = 1 \\ 0 & \text{otherwise} \end{cases}$$



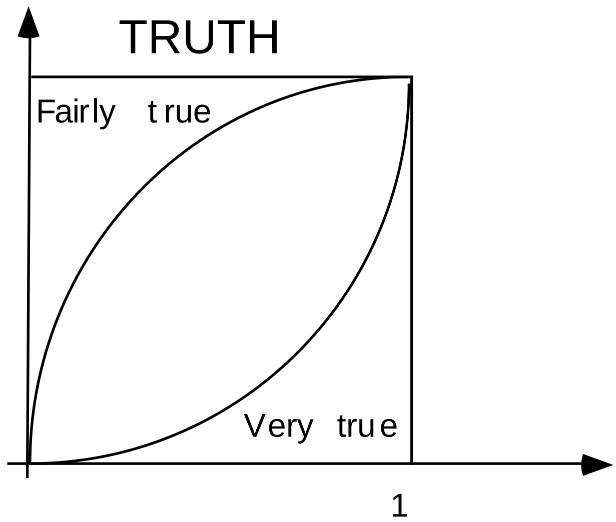
The world "Fairly" is interpreted as "more or less".

$$Fairly\ true(u) = \sqrt{u}$$

for each $u \in [0, 1]$.

$$Very\ true(u) = u^2$$

for each $u \in [0, 1]$.



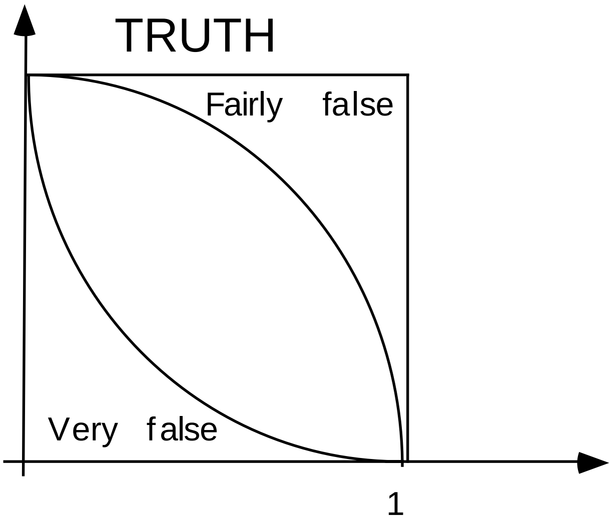
The world ”Fairly” is interpreted as ”more or less”.

$$Fairly\ false(u) = \sqrt{1-u}$$

for each $u \in [0, 1]$.

$$Very\ false(u) = (1-u)^2$$

for each $u \in [0, 1]$.



Suppose we have the fuzzy statement "x is A". Let τ be a term of linguistic variable **Truth**.

Then the statement "x is A is τ" is interpreted as "x is $\tau \circ A$". Where

$$(\tau \circ A)(u) = \tau(A(u))$$

for each $u \in [0, 1]$.

For example, let τ = "true". Then

"x is A is true"

is defined by

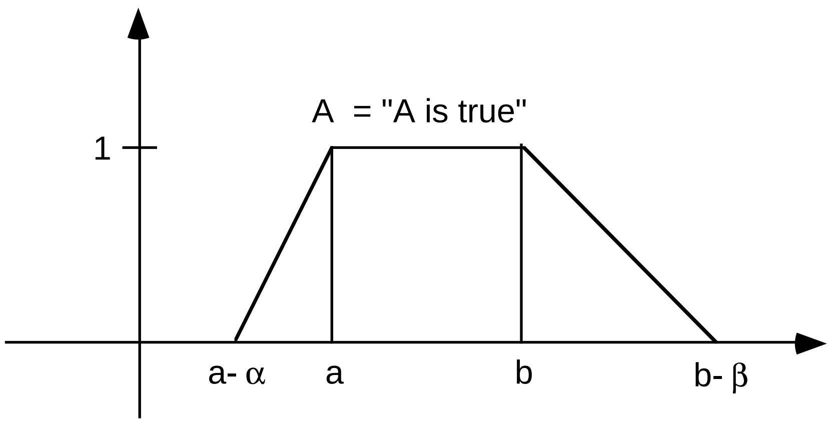
$$\text{"}x \text{ is } \tau \circ A\text{" = "}x \text{ is } A\text{"}$$

because

$$(\tau \circ A)(u) = \tau(A(u)) = A(u)$$

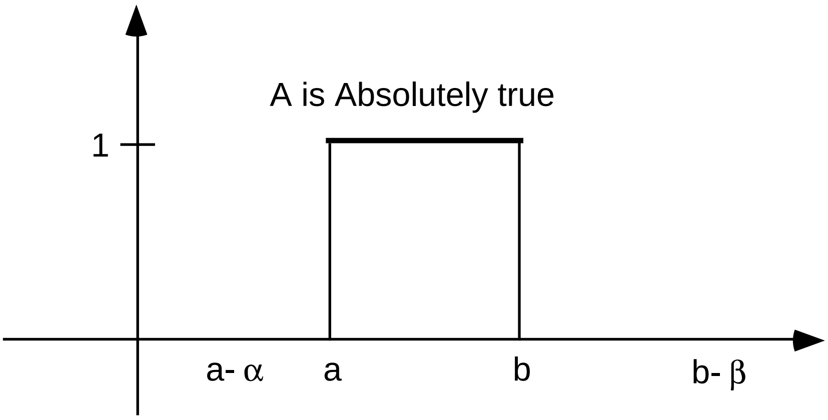
for each $u \in [0, 1]$.

It is why "everything we write is considered to be true".



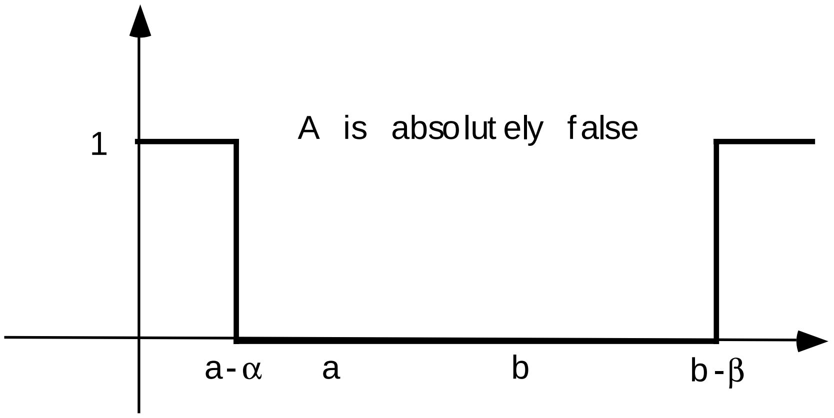
Let τ = "absolutely true". Then the statement "x is A is Absolutely true" is defined by "x is $\tau \circ A$", where

$$(\tau \circ A)(x) = \begin{cases} 1 & \text{if } A(x) = 1 \\ 0 & \text{otherwise} \end{cases}$$



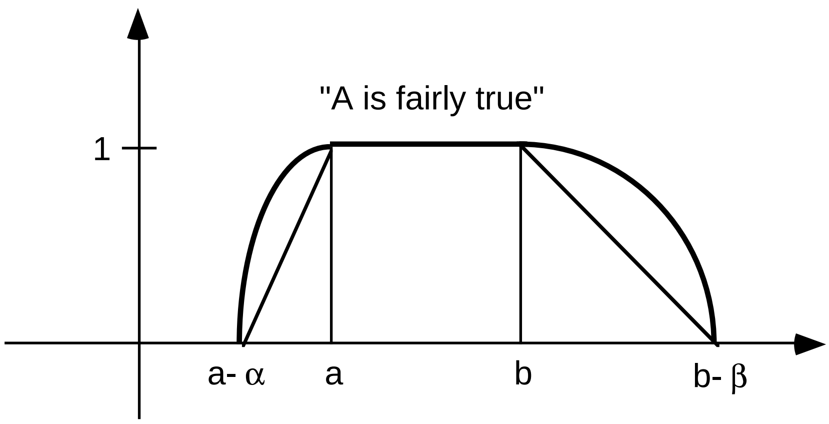
Let τ = "absolutely false". Then the statement "x is A is Absolutely false" is defined by "x is $\tau \circ A$", where

$$(\tau \circ A)(x) = \begin{cases} 1 & \text{if } A(x) = 0 \\ 0 & \text{otherwise} \end{cases}$$



Let τ = ”Fairly true”. Then the statement ”x is A is Fairly true” is defined by ”x is $\tau \circ A$”, where

$$(\tau \circ A)(x) = \sqrt{A(x)}$$



Let τ = ”Very true”. Then the statement ”x is A is

Fairly true” is defined by ”x is $\tau \circ A$”, where

$$(\tau \circ A)(x) = (A(x))^2$$