FS II: Fuzzy relations

A classical relation can be considered as a set of tuples, where a tuple is an ordered pair. A binary tuple is denoted by (u, v), an example of a ternary tuple is (u, v, w) and an example of n-ary tuple is $(x_1, \dots, x_n)$.

Example 1. *Let X be the domain of man $\{$John, Charles, James$\}$ and Y the domain of women $\{$Diana, Rita, Eva$\}$, then the realtion "married to" on $X \times Y$ is, for example*

$\{$*(Charles, Diana), (John, Eva), (James, Rita)* $\}$

Definition 1. *(classical n-ary relation) Let $X_1, \dots, X_n$ be classical sets. The subsets of the Cartesian product $X_1 \times \cdots \times X_n$ are called n-ary relations. If $X_1 = \cdots = X_n$ and $R \subset X^n$ then R is called an n-ary relation in X.*

Let R be a binary relation in $\mathbb{R}$. Then the charac-

teristic function of R is defined as

$$\chi_R(u,v) = \begin{cases} 1 & \textit{if } (u,v) \in R \\ 0 & otherwise \end{cases}$$

Example 2. *Consider the following relation*

$$(u,v) \in R \iff u \in [a,b] \textit{ and } v \in [0,c]$$

$$\chi_R(u,v) = \begin{cases} 1 & \textit{if } (u,v) \in [a,b] \times [0,c] \\ 0 & otherwise \end{cases}$$

Let R be a binary relation in a classical set X. Then

Definition 2. *(reflexivity) R is reflexive if* $\forall u \in U$: $(u,u) \in R$

Definition 3. *(anti-reflexivity) R is anti-reflexive if* $\forall u \in U : (u,u) \notin R$

Definition 4. *(symmetricity) R is symmetric if from* $(u,v) \in R \rightarrow (v,u) \in R,\ \forall u,v \in U$

Definition 5. *(anti-symmetricity) R is anti-symmetric if $(u,v) \in R$ and $(v,u) \in R$ then $u = v,\ \forall u,v \in U$*

Definition 6. *(transitivity) R is transitive if $(u,v) \in R$ and $(v,w) \in R$ then $(u,w) \in R,\ \forall u,v,w \in U$*

Example 3. *Consider the classical inequality relations on the real line $\mathbb{R}$. It is clear that $\leq$ is reflexive, anti-symmetric and transitive, $<$ is anti-reflexive, anti-symmetric and transitive.*

Other important properties of binary relations are

Property 1. *(equivalence) R is an equivalence relation if, R is reflexive, symmetric and transitive*

Property 2. *(partial order) R is a partial order relation if it is reflexive, anti-symmetric and transitive*

Property 3. *(total order) R is a total order relation if it is partial order and $\forall u, v \in R,\ (u, v) \in R$ or $(v, u) \in R$ hold*

Example 4. *Let us consider the binary relation "subset of". It is clear that we have a partial order relation.*

The relation $\leq$ on natural numbers is a total order relation.

Consider the relation "mod 3" on natural numbers

$$\{(m, n) \mid (n - m) \text{ mod } 3 \ \equiv 0\}$$

This is an equivalence relation.

Definition 7. *Let X and Y be nonempty sets. A fuzzy relation R is a fuzzy subset of $X \times Y$.*

In other words, $R \in \mathcal{F}(X \times Y)$.

If $X = Y$ then we say that R is a binary fuzzy relation in X.

Let R be a binary fuzzy relation on $\mathbb{R}$. Then $R(u, v)$ is interpreted as the degree of membership of the ordered pair (u, v) in R.

Example 5. *A simple example of a binary fuzzy relation on*

$$U = \{1, 2, 3\},$$

called "approximately equal" can be defined as

$$R(1, 1) = R(2, 2) = R(3, 3) = 1$$

$$R(1, 2) = R(2, 1) = R(2, 3) = R(3, 2) = 0.8$$

$$R(1, 3) = R(3, 1) = 0.3$$

The membership function of R is given by

$$R(u, v) = \begin{cases} 1 & \textit{if } u = v \\ 0.8 & \textit{if } |u - v| = 1 \\ 0.3 & \textit{if } |u - v| = 2 \end{cases}$$

In matrix notation it can be represented as

$$\begin{pmatrix} & 1 & 2 & 3 \\ 1 & 1 & 0.8 & 0.3 \\ 2 & 0.8 & 1 & 0.8 \\ 3 & 0.3 & 0.8 & 1 \end{pmatrix}$$

Operations on fuzzy relations

Fuzzy relations are very important because they can describe interactions between variables. Let R and S be two binary fuzzy relations on $X \times Y$.

Definition 8. *The intersection of R and S is defined by*

$$(R \wedge S)(u, v) = \min\{R(u, v), S(u, v)\}.$$

Note that $R\colon X \times Y \to [0, 1]$, i.e. R the domain of R is the whole Cartesian product $X \times Y$.

Definition 9. *The union of R and S is defined by*

$$(R \vee S)(u, v) = \max\{R(u, v), S(u, v)\}$$

Example 6. *Let us define two binary relations*

$$R = \text{"x is considerable larger than y"}$$

$$= \begin{pmatrix} & y_1 & y_2 & y_3 & y_4 \\ x_1 & 0.8 & 0.1 & 0.1 & 0.7 \\ x_2 & 0 & 0.8 & 0 & 0 \\ x_3 & 0.9 & 1 & 0.7 & 0.8 \end{pmatrix}$$

$$S = \text{"x is very close to y"}$$

$$= \begin{pmatrix} & y_1 & y_2 & y_3 & y_4 \\ x_1 & 0.4 & 0 & 0.9 & 0.6 \\ x_2 & 0.9 & 0.4 & 0.5 & 0.7 \\ x_3 & 0.3 & 0 & 0.8 & 0.5 \end{pmatrix}$$

The intersection of R and S means that "x is considerable larger than y" **and** *"x is very close to y".*

$$(R \wedge S)(x,y) = \begin{pmatrix} & y_1 & y_2 & y_3 & y_4 \\ x_1 & 0.4 & 0 & 0.1 & 0.6 \\ x_2 & 0 & 0.4 & 0 & 0 \\ x_3 & 0.3 & 0 & 0.7 & 0.5 \end{pmatrix}$$

The union of R and S means that ”x is considerable larger than y” **or** *”x is very close to y”.*

$$(R \vee S)(x,y) = \begin{pmatrix} & y_1 & y_2 & y_3 & y_4 \\ x_1 & 0.8 & 0 & 0.9 & 0.7 \\ x_2 & 0.9 & 0.8 & 0.5 & 0.7 \\ x_3 & 0.9 & 1 & 0.8 & 0.8 \end{pmatrix}$$

Consider a classical relation R on $\mathbb{R}$.

$$R(u,v) = \begin{cases} 1 & \text{if } (u,v) \in [a,b] \times [0,c] \\ 0 & otherwise \end{cases}$$

It is clear that the *projection* (or **shadow**) of R on the X-axis is the closed interval $[a,b]$ and its projection on the Y-axis is $[0,c]$.

If R is a classical relation in $X \times Y$ then

$$\Pi_X = \{x \in X \mid \exists y \in Y : (x,y) \in R\}$$

$$\Pi_Y = \{y \in Y \mid \exists x \in X : (x,y) \in R\}$$

where Π_X denotes projection on X and Π_Y denotes projection on Y.

Definition 10. *Let R be a fuzzy binary fuzzy relation on $X \times Y$. The projection of R on X is defined as*

$$\Pi_X(x) = \sup\{R(x, y) \mid y \in Y\}$$

and the projection of R on Y is defined as

$$\Pi_Y(y) = \sup\{R(x, y) \mid x \in X\}$$

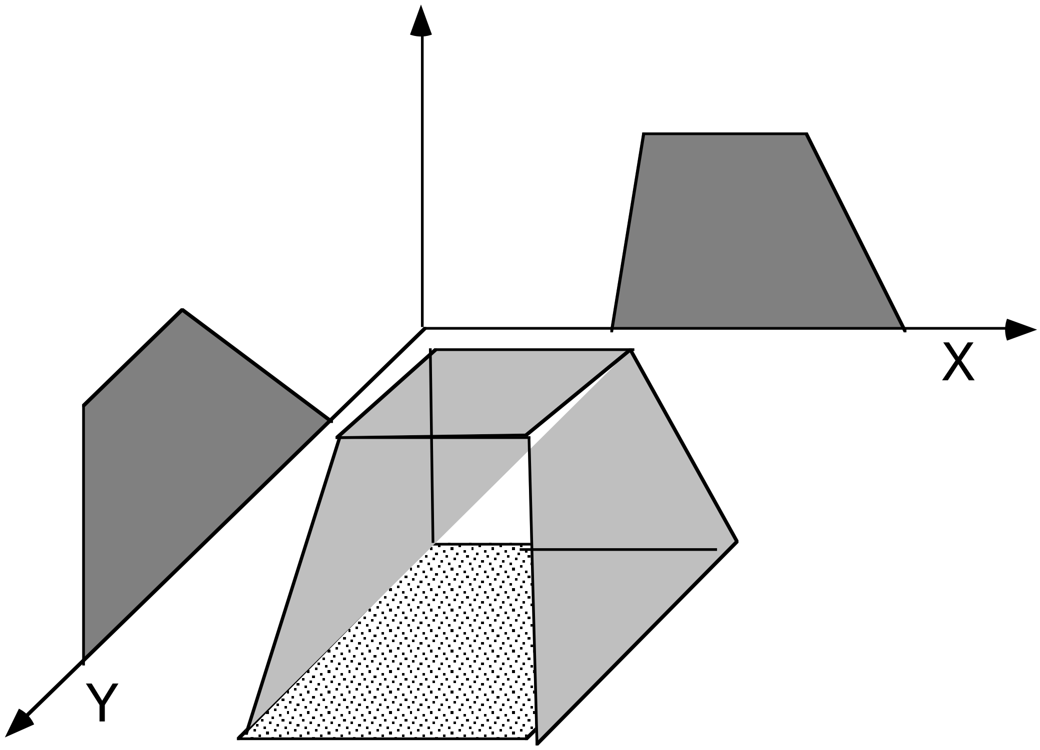
Example 7. *Consider the relation*

$R =$ *"x is considerable larger than y"*

$$= \begin{pmatrix} & y_1 & y_2 & y_3 & y_4 \\ x_1 & 0.8 & 0.1 & 0.1 & 0.7 \\ x_2 & 0 & 0.8 & 0 & 0 \\ x_3 & 0.9 & 1 & 0.7 & 0.8 \end{pmatrix}$$

then the projection on X means that

- x_1 *is assigned the highest membership degree from the tuples* (x_1, y_1), (x_1, y_2), (x_1, y_3), (x_1, y_4), *i.e.* $\Pi_X(x_1) = 1$, *which is the maximum of the first row.*
- x_2 *is assigned the highest membership degree from the tuples* (x_2, y_1), (x_2, y_2), (x_2, y_3), (x_2, y_4), *i.e.* $\Pi_X(x_2) = 0.8$, *which is the maximum of the second row.*
- x_3 *is assigned the highest membership degree from the tuples* (x_3, y_1), (x_3, y_2), (x_3, y_3), (x_3, y_4), *i.e.* $\Pi_X(x_3) = 1$, *which is the maximum of the third row.*

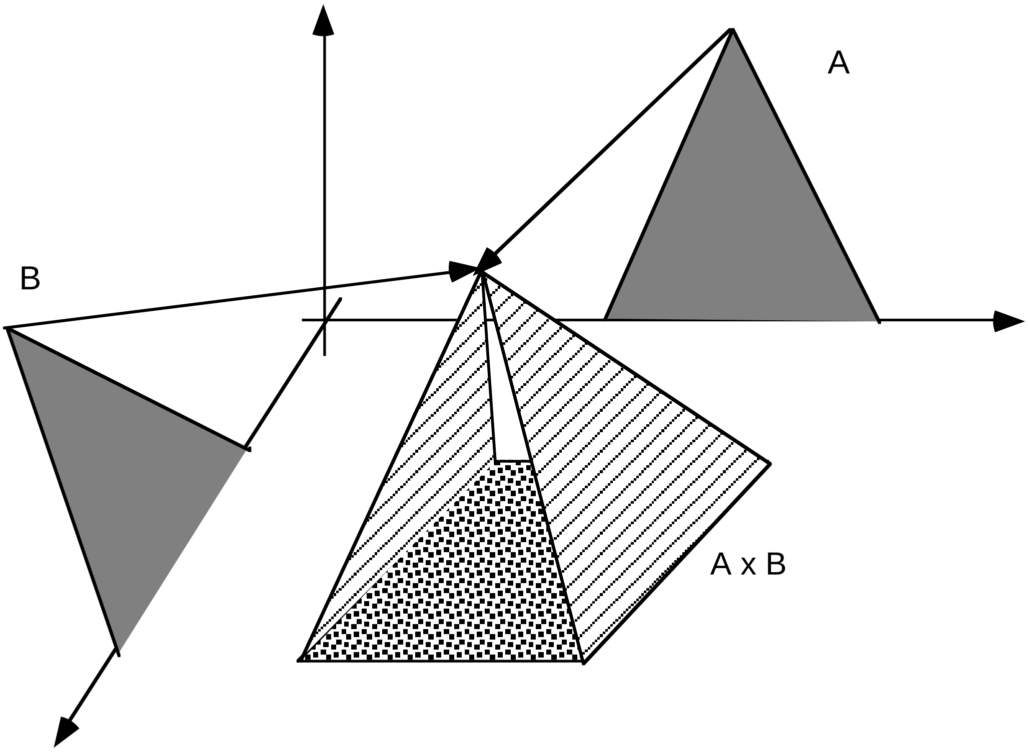


Shadows of a fuzzy relation.

Definition 11. *The Cartesian product of* $A \in \mathcal{F}(X)$ *and* $B \in \mathcal{F}(Y)$ *is defined as*

$$(A \times B)(u, v) = \min\{A(u), B(v)\}.$$

for all $u \in X$ *and* $v \in Y$.



It is clear that the Cartesian product of two fuzzy sets is a fuzzy relation in $X \times Y$.

If A and B are normal then $\Pi_Y(A \times B) = B$ and $\Pi_X(A \times B) = A$.

Really,

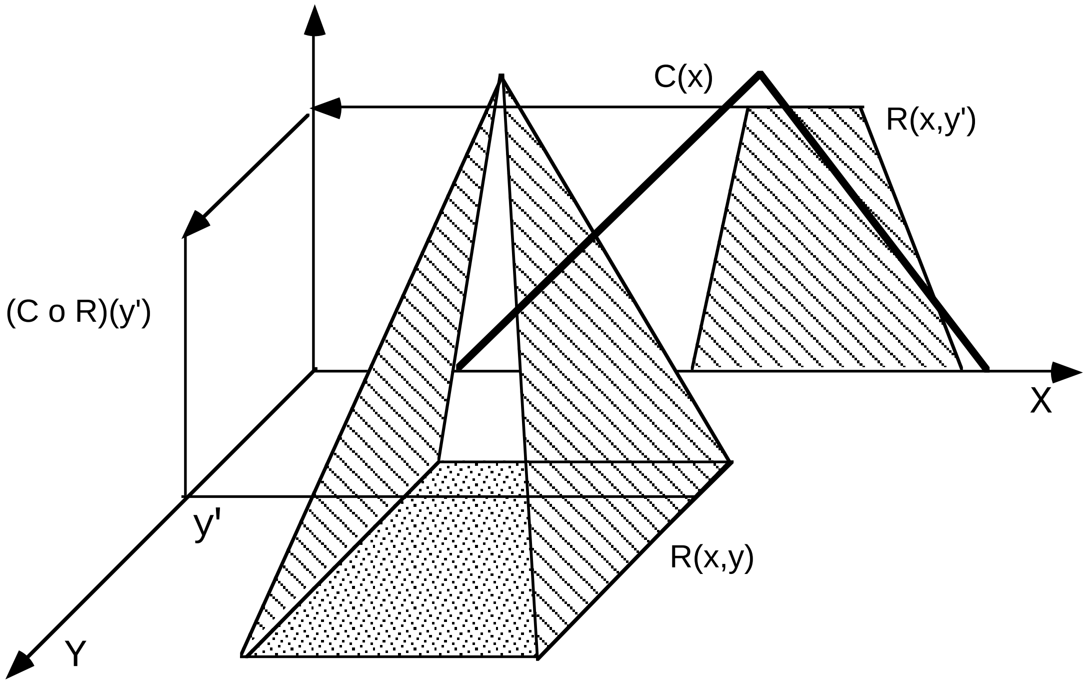
$$\Pi_X(x) = \sup\{(A \times B)(x, y) \mid y\}$$
$$= \sup\{A(x) \wedge B(y) \mid y\} =$$
$$\min\{A(x), \sup\{B(y)\} \mid y\}$$
$$= \min\{A(x), 1\} = A(x).$$

Definition 12. *The sup-min composition of a fuzzy set $C \in \mathcal{F}(X)$ and a fuzzy relation $R \in \mathcal{F}(X \times Y)$ is defined as*

$$(C \circ R)(y) = \sup_{x \in X} \min\{C(x), R(x, y)\}$$

for all $y \in Y$.

The composition of a fuzzy set C and a fuzzy relation R can be considered as the shadow of the relation R on the fuzzy set C.



Example 8. *Let A and B be fuzzy numbers and let*

$$R = A \times B$$

a fuzzy relation.

Observe the following property of composition

$$A \circ R = A \circ (A \times B) = A,$$

$$B \circ R = B \circ (A \times B) = B.$$

Example 9. *Let* C *be a fuzzy set in the universe of discourse* $\{1, 2, 3\}$ *and let* R *be a binary fuzzy relation in* $\{1, 2, 3\}$. *Assume that*

$$C = 0.2/1 + 1/2 + 0.2/3$$

and

$$R = \begin{pmatrix} & 1 & 2 & 3 \\ 1 & 1 & 0.8 & 0.3 \\ 2 & 0.8 & 1 & 0.8 \\ 3 & 0.3 & 0.8 & 1 \end{pmatrix}$$

Using the definition of sup-min composition we get

$$C \circ R = (0.2/1 + 1/2 + 0.2/3) \circ \begin{pmatrix} & 1 & 2 & 3 \\ 1 & 1 & 0.8 & 0.3 \\ 2 & 0.8 & 1 & 0.8 \\ 3 & 0.3 & 0.8 & 1 \end{pmatrix} =$$

$$0.8/1 + 1/2 + 0.8/3.$$

Example 10. *Let C be a fuzzy set in the universe of discourse $[0, 1]$ and let R be a binary fuzzy relation in $[0, 1]$. Assume that $C(x) = x$ and*

$$R(x, y) = 1 - |x - y|.$$

Using the definition of sup-min composition we get

$$(C \circ R)(y) = \sup_{x \in [0,1]} \min\{x, 1 - |x - y|\} = \frac{1 + y}{2}$$

for all $y \in [0, 1]$.

Definition 13. *(sup-min composition of fuzzy relations) Let $R \in \mathcal{F}(X \times Y)$ and $S \in \mathcal{F}(Y \times Z)$. The sup-min composition of R and S, denoted by $R \circ S$ is defined as*

$$(R \circ S)(u, w) = \sup_{v \in Y} \min\{R(u, v), S(v, w)\}$$

It is clear that $R \circ S$ is a binary fuzzy relation in $X \times Z$.

Example 11. *Consider two fuzzy relations*

$R =$ *"x is considerable larger than y"*

$$= \begin{pmatrix} & y_1 & y_2 & y_3 & y_4 \\ x_1 & 0.8 & 0.1 & 0.1 & 0.7 \\ x_2 & 0 & 0.8 & 0 & 0 \\ x_3 & 0.9 & 1 & 0.7 & 0.8 \end{pmatrix}$$

$$S = \text{"y is very close to z"} = \begin{pmatrix} & z_1 & z_2 & z_3 \\ y_1 & 0.4 & 0.9 & 0.3 \\ y_2 & 0 & 0.4 & 0 \\ y_3 & 0.9 & 0.5 & 0.8 \\ y_4 & 0.6 & 0.7 & 0.5 \end{pmatrix}$$

Then their composition is

$$R \circ S = \begin{pmatrix} & z_1 & z_2 & z_3 \\ x_1 & 0.6 & 0.8 & 0.5 \\ x_2 & 0 & 0.4 & 0 \\ x_3 & 0.7 & 0.9 & 0.7 \end{pmatrix}$$

Formally,

$$\begin{pmatrix} & y_1 & y_2 & y_3 & y_4 \\ x_1 & 0.8 & 0.1 & 0.1 & 0.7 \\ x_2 & 0 & 0.8 & 0 & 0 \\ x_3 & 0.9 & 1 & 0.7 & 0.8 \end{pmatrix} \circ \begin{pmatrix} & z_1 & z_2 & z_3 \\ y_1 & 0.4 & 0.9 & 0.3 \\ y_2 & 0 & 0.4 & 0 \\ y_3 & 0.9 & 0.5 & 0.8 \\ y_4 & 0.6 & 0.7 & 0.5 \end{pmatrix} =$$

$$\begin{pmatrix} & z_1 & z_2 & z_3 \\ x_1 & 0.6 & 0.8 & 0.5 \\ x_2 & 0 & 0.4 & 0 \\ x_3 & 0.7 & 0.9 & 0.7 \end{pmatrix}$$

i.e., the composition of R *and* S *is nothing else, but the classical product of the matrices* R *and* S *with the difference that instead of addition we use maximum and instead of multiplication we use minimum operator.*