FS I: Fuzzy Sets and Fuzzy Logic

Fuzzy sets were introduced by Zadeh in 1965 to represent/manipulate data and information possessing nonstatistical uncertainties.

- L.A.Zadeh, Fuzzy Sets, *Information and Control*, 8(1965) 338-353.

It was specifically designed to mathematically represent uncertainty and vagueness and to provide formalized tools for dealing with the imprecision intrinsic to many problems. However, the story of fuzzy logic started much more earlier . . .

To devise a concise theory of logic, and later mathematics, *Aristotle* posited the so-called ”Laws of Thought”.

One of these, the ”Law of the Excluded Middle,” states that every proposition must either be *True* (**T**) or *False* (**F**). Even when *Parminedes* proposed the first version of this law (around 400 Before Christ) there were strong and immediate objections: for example, *Heraclitus* proposed that things could be simultaneously *True* and *not True*.

It was *Plato* who laid the foundation for what would become fuzzy logic, indicating that there was a third region (beyond **T** and **F**) where these opposites ”tumbled about.” A systematic alternative to the bi-valued logic of Aristotle was first proposed by *Łukasiewicz* around 1920, when he described a three-valued logic, along with the mathematics to accompany it. The third value he proposed can best be translated as the term ”possible,” and he assigned it a numeric value between **T** and **F**. Eventually, he proposed an entire notation and axiomatic system from which he hoped to derive modern mathematics.

Later, he explored four-valued logics, five-valued logics, and then declared that in principle there was nothing to prevent the derivation of an infinite-valued logic.Łukasiewicz felt that three- and infinite-valued logics were the most intriguing, but he ultimately settled on a four-valued logic because it seemed to be the most easily adaptable to Aristotelian logic.

It should be noted that *Knuth* also proposed a three-valued logic similar to Lukasiewicz's, from which he speculated that mathematics would become even more elegant than in traditional bi-valued logic.

The notion of an infinite-valued logic was introduced in Zadeh's seminal work "Fuzzy Sets" where he described the mathematics of fuzzy set theory, and by extension fuzzy logic. This theory proposed making the membership function (or the values **F** and **T**) operate over the range of real numbers [0, 1]. New operations for the calculus of logic were proposed, and showed to be in principle at least a generalization of classic logic.

Fuzzy logic provides an inference morphology that enables approximate human reasoning capabilities to be applied to knowledge-based systems. The theory of fuzzy logic provides a mathematical strength to capture the uncertainties associated with human cognitive processes, such as thinking and reasoning.

The conventional approaches to knowledge representation lack the means for representating the meaning of fuzzy concepts. As a consequence, the approaches based on first order logic and classical probablity theory do not provide an appropriate conceptual framework for dealing with the representation of commonsense knowledge, since such knowledge is by its nature both lexically imprecise and noncategorical.

The developement of fuzzy logic was motivated in large measure by the need for a conceptual framework which can address the issue of uncertainty and lexical imprecision.

Some of the essential characteristics of fuzzy logic relate to the following (Zadeh, 1992):

- In fuzzy logic, exact reasoning is viewed as a limiting case of approximate reasoning.
- In fuzzy logic, everything is a matter of degree.
- In fuzzy logic, knowledge is interpreted a collection of elastic or, equivalently, fuzzy constraint on a collection of variables.
- Inference is viewed as a process of propagation of elastic constraints.
- Any logical system can be fuzzified.

There are two main characteristics of fuzzy systems that give them better performance for specific applications.

- Fuzzy systems are suitable for uncertain or approximate reasoning, especially for the system with a mathematical model that is difficult to derive.
- Fuzzy logic allows decision making with estimated values under incomplete or uncertain information.

Definition 1. *(fuzzy set) Let X be a nonempty set. A fuzzy set A in X is characterized by its membership function*

$$\mu_A\colon X \to [0,1]$$

and $\mu_A(x)$ is interpreted as the degree of membership of element x in fuzzy set A for each $x \in X$.

It is clear that A is completely determined by the set of tuples

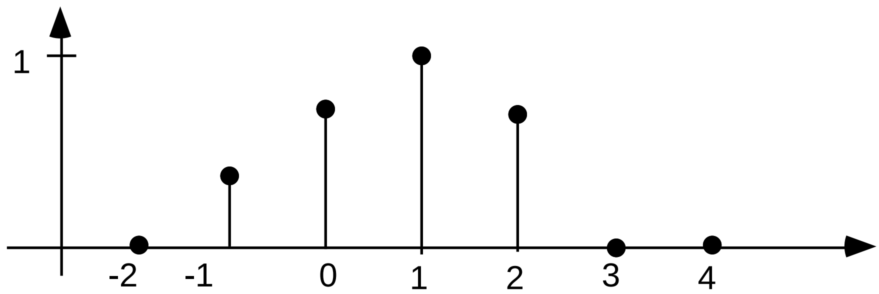
$$A = \{(u, \mu_A(u)) | u \in X\}.$$

Frequently we will write $A(x)$ instead of $\mu_A(x)$. The family of all fuzzy sets in X is denoted by $\mathcal{F}(X)$.

If $X = \{x_1, \ldots, x_n\}$ is a finite set and A is a fuzzy set in X then we often use the notation

$$A = \mu_1/x_1 + \cdots + \mu_n/x_n$$

where the term $\mu_i/x_i,\ i = 1, \ldots, n$ signifies that μ_i is the grade of membership of x_i in A and the plus sign represents the union.

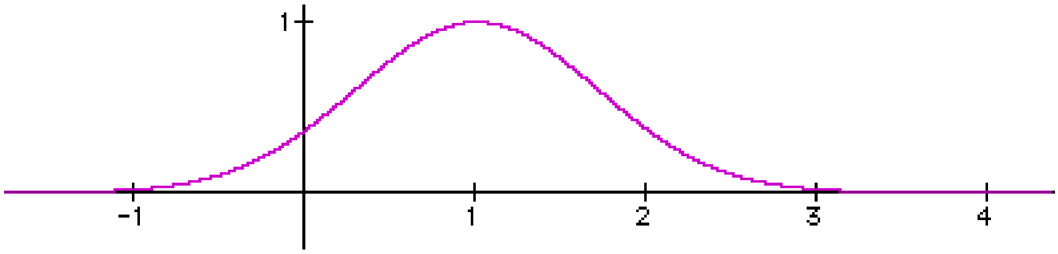


A discrete membership function for ”x is close to 1”.

Example 1. *The membership function of the fuzzy set of* real *numbers ”close to 1”, is can be defined* *as*

$$A(t) = \exp(-\beta(t-1)^2)$$

where β is a positive real number.

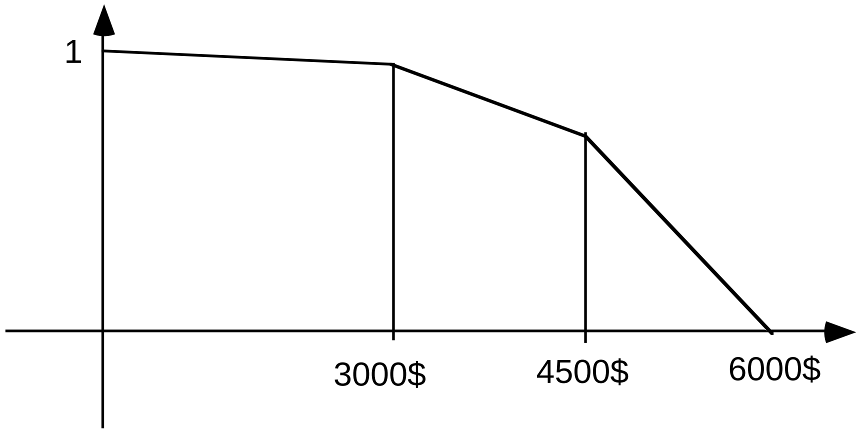


A membership function for ”x is close to 1”.

Example 2. *Assume someone wants to buy a cheap car.* Cheap *can be represented as a fuzzy set on a universe of prices, and depends on his purse. For instance, from the Figure* cheap *is roughly interpreted as follows:*

- *Below 3000$ cars are considered as cheap, and prices make no real difference to buyer's eyes.*
- *Between 3000$ and 4500$, a variation in the price induces a weak preference in favor of the cheapest car.*

- *Between 4500\$ and 6000\$, a small variation in the price induces a clear preference in favor of the cheapest car.*
- *Beyond 6000\$ the costs are too high (out of consideration).*



Membership function of ”cheap”.

Definition 2. *(support) Let A be a fuzzy subset of X; the* support *of A, denoted $supp(A)$, is the crisp subset of X whose elements all have nonzero membership grades in A.*

$$supp(A) = \{x \in X | A(x) > 0\}.$$

Definition 3. *(normal fuzzy set) A fuzzy subset A of a classical set X is called* normal *if there exists an*

$x \in X$ *such that* $A(x) = 1$. *Otherwise* A *is subnormal.*

Definition 4. *(α-cut) An α-level set of a fuzzy set* A *of* X *is a non-fuzzy set denoted by* $[A]^\alpha$ *and is defined by*

$$[A]^\alpha = \begin{cases} \{t \in X | A(t) \geq \alpha\} & \textit{if } \alpha > 0 \\ cl(\textit{supp}A) & \textit{if } \alpha = 0 \end{cases}$$

where $cl(\textit{supp}A)$ *denotes the closure of the support of* A.

Example 3. *Assume* $X = \{-2, -1, 0, 1, 2, 3, 4\}$ *and*

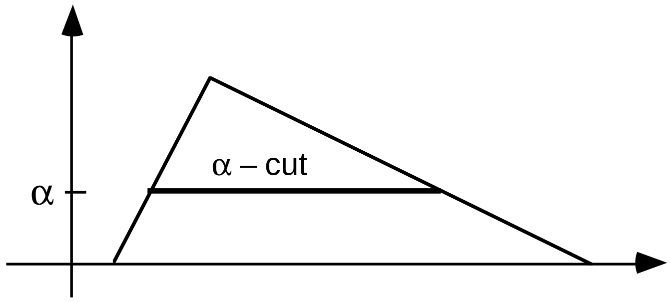
$$A = 0.0/-2 + 0.3/-1$$

$$+0.6/0 + 1.0/1 + 0.6/2 + 0.3/3 + 0.0/4,$$

in this case

$$[A]^\alpha = \begin{cases} \{-1, 0, 1, 2, 3\} & \textit{if } 0 \leq \alpha \leq 0.3 \\ \{0, 1, 2\} & \textit{if } 0.3 < \alpha \leq 0.6 \\ \{1\} & \textit{if } 0.6 < \alpha \leq 1 \end{cases}$$

Definition 5. *(convex fuzzy set) A fuzzy set* A *of* X *is called* convex *if* $[A]^\alpha$ *is a convex subset of* X $\forall \alpha \in [0, 1]$.



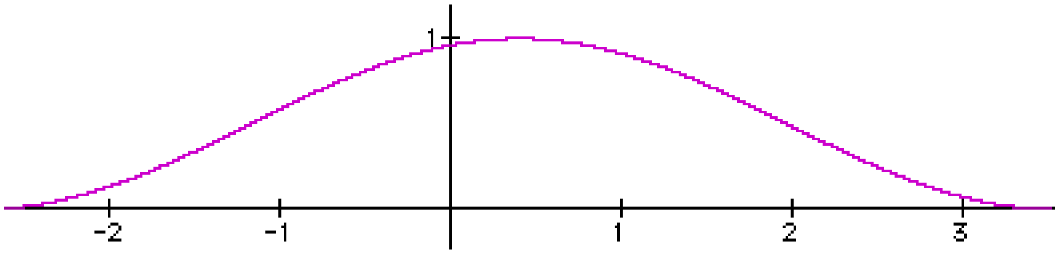
An α-cut of a triangular fuzzy number.

In many situations people are only able to characterize numeric information imprecisely. For example, people use terms such as, about 5000, near zero, or essentially bigger than 5000. These are examples of what are called *fuzzy numbers*. Using the theory of fuzzy subsets we can represent these fuzzy numbers as fuzzy subsets of the set of real numbers. More exactly,

Definition 6. *(fuzzy number) A fuzzy number A is a fuzzy set of the real line with a normal, (fuzzy) convex and continuous membership function of bounded support. The family of fuzzy numbers will be denoted by $\mathcal{F}$.*

Definition 7. *(quasi fuzzy number) A quasi fuzzy number A is a fuzzy set of the real line with a normal, fuzzy convex and continuous membership function satisfying the limit conditions*

$$\lim_{t\to\infty} A(t) = 0, \quad \lim_{t\to-\infty} A(t) = 0.$$



Fuzzy number.

Let A be a fuzzy number. Then $[A]^\gamma$ is a closed convex (compact) subset of $\mathbb{R}$ for all $\gamma \in [0,1]$. Let us introduce the notations

$$a_1(\gamma) = \min[A]^\gamma, \quad a_2(\gamma) = \max[A]^\gamma$$

In other words, $a_1(\gamma)$ denotes the left-hand side and $a_2(\gamma)$ denotes the right-hand side of the γ-cut. It is easy to see that

$$\text{If } \alpha \le \beta \text{ then } [A]^\alpha \supset [A]^\beta$$

Furthermore, the left-hand side function

$$a_1 \colon [0, 1] \to \mathbb{R}$$

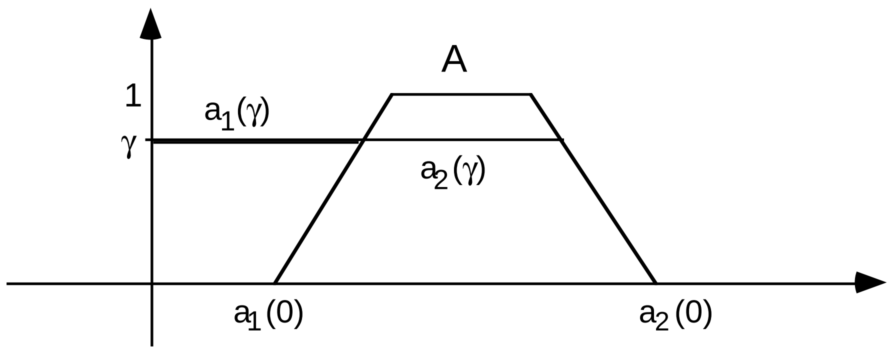
is monoton increasing and lower semicontinuous, and the right-hand side function

$$a_2 \colon [0, 1] \to \mathbb{R}$$

is monoton decreasing and upper semicontinuous. We shall use the notation

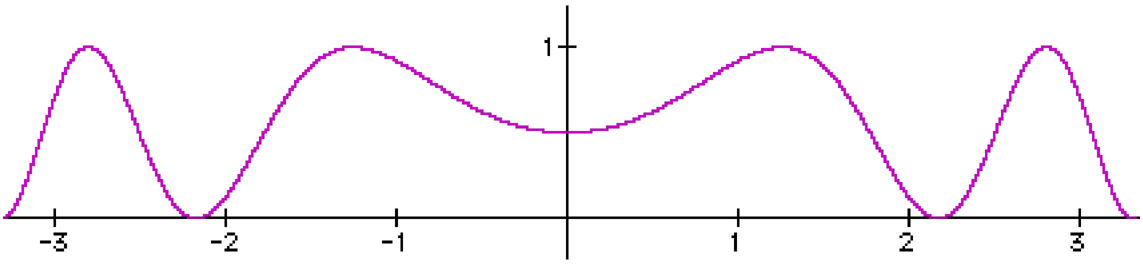
$$[A]^\gamma = [a_1(\gamma), a_2(\gamma)].$$

The support of A is the open interval $(a_1(0), a_2(0))$.



The support of A is $(a_1(0), a_2(0))$.

If A is not a fuzzy number then there exists an $\gamma \in [0, 1]$ such that $[A]^\gamma$ is not a convex subset of $\mathbb{R}$.



Not fuzzy number.

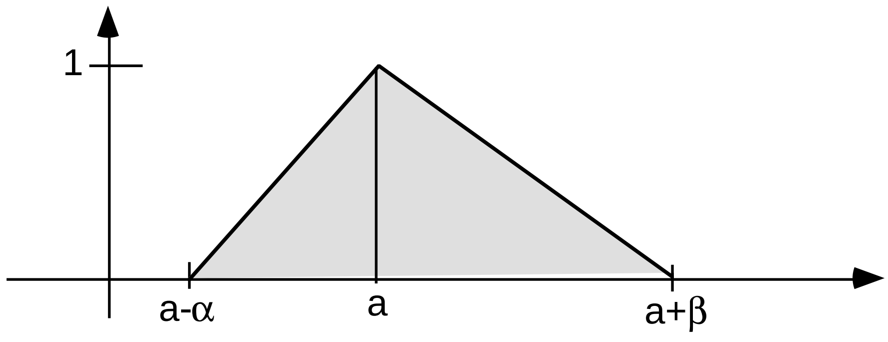
Definition 8. *(triangular fuzzy number) A fuzzy set A is called triangular fuzzy number with peak (or center) a, left width $\alpha > 0$ and right width $\beta > 0$ if its membership function has the following form*

$$A(t) = \begin{cases} 1 - \dfrac{a-t}{\alpha} & \textit{if } a - \alpha \le t \le a \\ 1 - \dfrac{t-a}{\beta} & \textit{if } a \le t \le a + \beta \\ 0 & \textit{otherwise} \end{cases}$$

and we use the notation $A = (a, \alpha, \beta)$. It can easily be verified that

$$[A]^\gamma = [a - (1-\gamma)\alpha, a + (1-\gamma)\beta], \ \forall \gamma \in [0, 1].$$

The support of A is $(a - \alpha, b + \beta)$.



Triangular fuzzy number.

A triangular fuzzy number with center a may be seen as a fuzzy quantity

”x is approximately equal to a”.

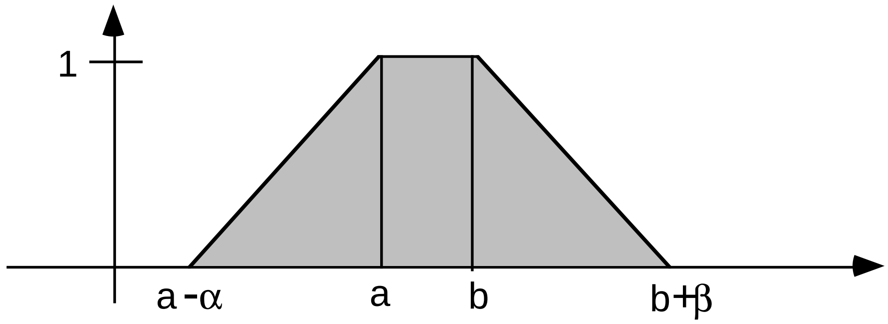
Definition 9. *(trapezoidal fuzzy number) A fuzzy set A is called trapezoidal fuzzy number with tolerance interval $[a, b]$, left width α and right width β if its membership function has the following form*

$$A(t) = \begin{cases} 1 - (a-t)/\alpha & \textit{if } a - \alpha \le t \le a \\ 1 & \textit{if } a \le t \le b \\ 1 - (t-b)/\beta & \textit{if } a \le t \le b + \beta \\ 0 & \textit{otherwise} \end{cases}$$

and we use the notation $A = (a, b, \alpha, \beta)$. *It can easily be shown that*

$$[A]^\gamma = [a - (1-\gamma)\alpha, b + (1-\gamma)\beta], \ \forall \gamma \in [0,1].$$

The support of A *is* $(a - \alpha, b + \beta)$.



Trapezoidal fuzzy number.

A trapezoidal fuzzy number may be seen as a fuzzy quantity

"x is approximately in the interval $[a, b]$".

Definition 10. *Any fuzzy number* $A \in \mathcal{F}$ *can be de-*

scribed as

$$A(t) = \begin{cases} L\left(\dfrac{a-t}{\alpha}\right) & \textit{if } t \in [a-\alpha, a] \\ 1 & \textit{if } t \in [a, b] \\ R\left(\dfrac{t-b)}{\beta}\right) & \textit{if } t \in [b, b+\beta] \\ 0 & \textit{otherwise} \end{cases}$$

where $[a, b]$ *is the* peak *or* core *of* A*,*

$$L\colon [0,1] \to [0,1], \quad R\colon [0,1] \to [0,1],$$

are continuous and non-increasing shape functions with $L(0) = R(0) = 1$ *and* $R(1) = L(1) = 0$*. We call this fuzzy interval of* LR*-type and refer to it by*

$$A = (a, b, \alpha, \beta)_{LR}.$$

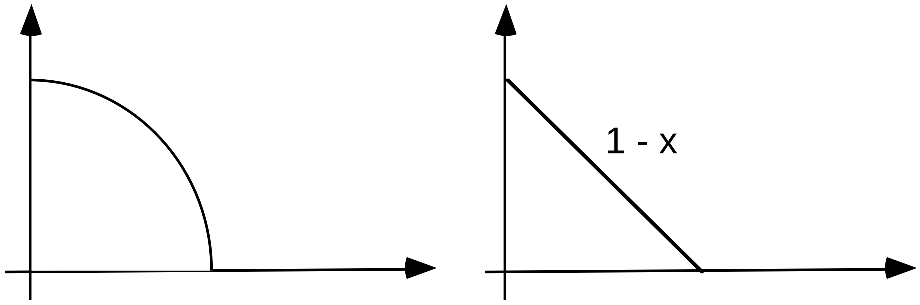
The support of A *is* $(a - \alpha, b + \beta)$*.*

Let $A = (a, b, \alpha, \beta)_{LR}$ be a fuzzy number of type LR. If $a = b$ then we use the notation

$$A = (a, \alpha, \beta)_{LR}$$

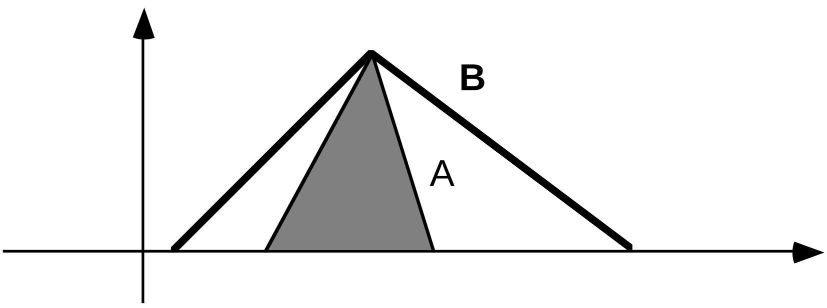
and say that A is a quasi-triangular fuzzy number. Furthermore if $L(x) = R(x) = 1 - x$ then instead of $A = (a, b, \alpha, \beta)_{LR}$ we simply write

$$A = (a, b, \alpha, \beta).$$



Nonlinear and linear reference functions.

Definition 11. *(subsethood) Let A and B are fuzzy subsets of a classical set X. We say that A is a subset of B if $A(t) \leq B(t)$, $\forall t \in X$.*



A is a subset of B.

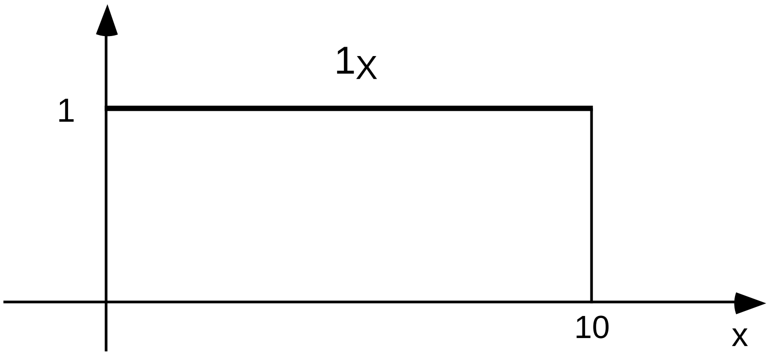
Definition 12. *(equality of fuzzy sets) Let A and B are fuzzy subsets of a classical set X. A and B are said to be equal, denoted $A = B$, if $A \subset B$ and $B \subset A$. We note that $A = B$ if and only if $A(x) = B(x)$ for $x \in X$.*

Definition 13. *(empty fuzzy set) The empty fuzzy subset of X is defined as the fuzzy subset $\emptyset$ of X such that $\emptyset(x) = 0$ for each $x \in X$.*

It is easy to see that $\emptyset \subset A$ holds for any fuzzy subset A of X.

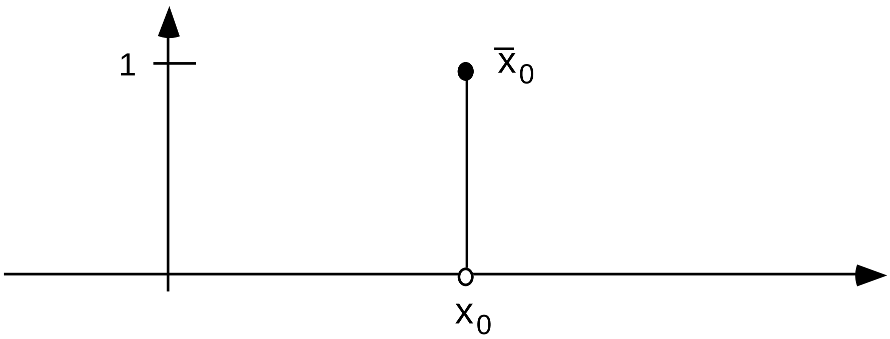
Definition 14. *The largest fuzzy set in X, called universal fuzzy set in X, denoted by 1_X, is defined by $1_X(t) = 1$, $\forall t \in X$.*

It is easy to see that $A \subset 1_X$ holds for any fuzzy subset A of X.



The graph of the universal fuzzy subset in $X = [0, 10]$.

Definition 15. *(Fuzzy point) Let A be a fuzzy number. If $supp(A) = \{x_0\}$ then A is called a fuzzy point and we use the notation $A = \bar{x}_0$.*



Fuzzy point.

Let $A = \bar{x}_0$ be a fuzzy point. It is easy to see that $[A]^\gamma = [x_0, x_0] = \{x_0\}, \ \forall \gamma \in [0, 1]$.

Operations on fuzzy sets

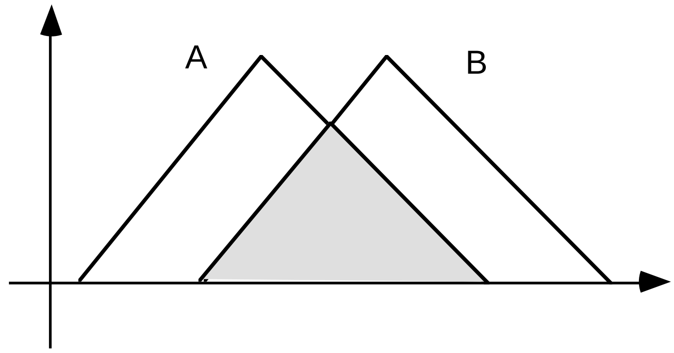
We extend the classical set theoretic operations from ordinary set theory to fuzzy sets. We note that all those operations which are extensions of crisp concepts reduce to their usual meaning when the fuzzy subsets have membership degrees that are drawn from $\{0, 1\}$. For this reason, when extending operations to fuzzy sets we use the same symbol as in set theory.

Let A and B are fuzzy subsets of a nonempty (crisp) set X.

Definition 16. *(intersection) The intersection of A and B is defined as*

$$(A \cap B)(t) = \min\{A(t), B(t)\} = A(t) \wedge B(t),$$

for all $t \in X$.

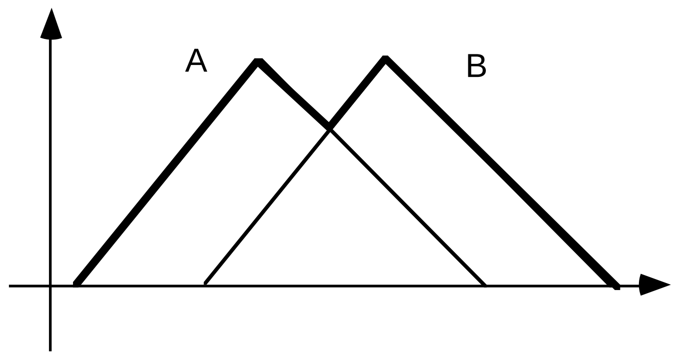


Intersection of two triangular fuzzy numbers.

Definition 17. *(union) The union of A and B is defined as*

$$(A \cup B)(t) = \max\{A(t), B(t)\} = A(t) \vee B(t),$$

for all $t \in X$.



Union of two triangular fuzzy numbers.

Definition 18. *(complement) The complement of a* *fuzzy set A is defined as*

$$(\neg A)(t) = 1 - A(t)$$

A closely related pair of properties which hold in ordinary set theory are *the law of excluded middle*

$$A \vee \neg A = X$$

and *the law of noncontradiction principle*

$$A \wedge \neg A = \emptyset$$

It is clear that $\neg 1_X = \emptyset$ and $\neg\emptyset = 1_X$, however, the laws of excluded middle and noncontradiction are not satisfied in fuzzy logic.

Lemma 1. *The law of excluded middle is not valid. Let $A(t) = 1/2,\ \forall t \in \mathbb{R}$, then it is easy to see that*

$$(\neg A \vee A)(t) = \max\{\neg A(t), A(t)\}$$

$$= \max\{1 - 1/2, 1/2\} =$$

$$1/2 \neq 1.$$

Lemma 2. *The law of noncontradiction is not valid. Let* $A(t) = 1/2,\ \forall t \in \mathbb{R}$*, then it is easy to see that*

$$(\neg A \wedge A)(t) = \min\{\neg A(t), A(t)\} =$$
$$\min\{1 - 1/2, 1/2\} =$$
$$1/2 \neq 0.$$

However, fuzzy logic does satisfy *De Morgan*'s laws

$$\neg(A \wedge B) = \neg A \vee \neg B,\ \neg(A \vee B) = \neg A \wedge \neg B.$$