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Fuzzy Approaches to Multiple Objective Programming - Tutorial

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1. Multiple objective programs

Consider a multiple objective program

$$\max_{x \in X} \{ f_1(x), \dots, f_k(x) \}$$

where

- $f_i \colon \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, k$ are objective functions,
- $\mathbb{R}^k$ is the *criterion space*,
- $x \in \mathbb{R}^n$ is the decision variable,
- $\mathbb{R}^n$ is the *decision space*, $X \subset \mathbb{R}^n$ is called the *set of feasible alternatives*.

The image of X in $\mathbb{R}^k$, denoted by Z_X, i.e. the set of *feasible outcomes* is defined as

$$Z_X = \{z \in \mathbb{R}^k | z_i = f_i(x), i = 1, \ldots, k, x \in X\}.$$

Definition 1.1. *An $x^* \in X$ is said to be efficient (or nondominated or Pareto-optimal) for the MOP iff there exists no $y \in X$ such that*

$$f_i(y) \geq f_i(x^*)$$

for all i with strict inequality for at least one i. The set of all Pareto-optimal solutions will be denoted by X^*.

In other words, a nondominated point is such that any other point in Z_X which increases the value of one criterion also decreases the value of at least one other criterion.

Example 1.1. *Suppose that we are given a three-objective decison problem*

$$\max_{x \in X} \{f_1(x), f_2(x), f_3(x)\},$$

where $X = \{u, y, z\}$ *is a finite set and let*

$$\{f_1(u), f_2(u), f_3(u)\} = (1, 3, 3)$$
$$\{f_1(y), f_2(y), f_3(y)\} = (2, 2, 3)$$
$$\{f_1(z), f_2(z), f_3(z)\} = (1, 2, 2)$$

be the set of feasible outcomes, i.e.

$$Z_X = \{(1, 3, 3), (2, 2, 3), (1, 2, 2)\}$$

Then u *and* y *are the efficient solutions.*

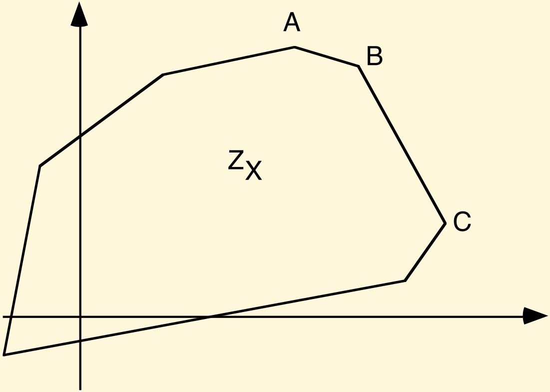

Figure 1: Segments AB and BC are Pareto-optimal.

The simplest bi-objective programming problem is,

$$\max\{x, 1-x\}; \text{ subject to } x \in [0, 1].$$

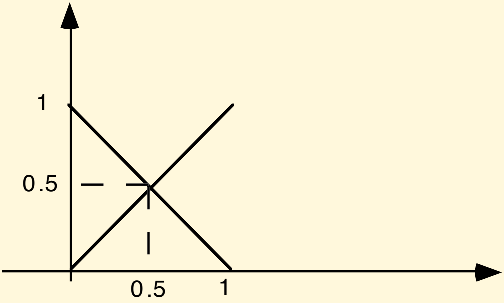

Figure 2: A simple two-objective problem.

Here we have $X^* = [0, 1]$, i.e. each feasible alternative is an efficient solution.

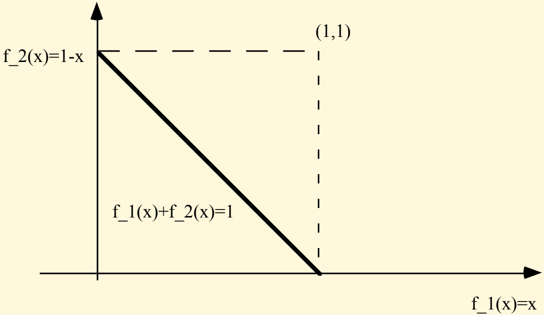

Figure 3: The set of Pareto-optimal solution is [0,1].

A natural way to obtain initial information about the decision problem is to optimize each criterion separately. Let x^i be a solution to

$$f_i(x^i) = \max\{f_i(x) | x \in X\},$$

and let

$$M_i = f_i(x^i),$$

be the optimal value of the i-th individual objective function over X.

The vector

$$M = (M_1, M_2, \dots, M_k)$$

is called the *ideal point*.

The ideal point for the bi-objective problem is $(1, 1)$ (which is not attainable by any point from the decision set $[0, 1]$).

The *payoff matrix* of an k-objective problem is defined as

$$\begin{pmatrix} M_1 & f_1(x^2) & \dots & f_1(x^k) \\ f_2(x^1) & M_2 & \dots & f_2(x^k) \\ \vdots & \vdots & & \vdots \\ f_k(x^1) & f_k(x^2) & \dots & M_k \end{pmatrix}$$

Here we have used the notation $M_i = f_i(x^i)$. The payoff matrix for the bi-objective problem is,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Let m_i denote the value

$$\min\{f_i(x)|x \in X\}$$

i.e. m_i is the worst possible value for the i-th objective.

It is clear that the inequalities

$$m_i \leq f_i(x) \leq M_i,$$

hold for each x in X.

In the case of the bi-objective problem we have

$$0 \leq f_1(x), f_2(x) \leq 1,$$

for all x from $X = [0, 1]$.

2. Application functions

An application function h_i for the MOP

$$\max_{x \in X} \{ f_1(x), \dots, f_k(x) \},$$

is defined as $h_i \colon \mathbb{R} \to [0, 1]$, where $h_i(t)$ measures the degree of fulfillment of the decision maker's requirements about the i-th objective by the value t. Suppose that the decision maker has some preference parameters, for example

- *reference points* which represents desirable levels on each criterion
- *reservation levels* which represent minimal requirements on each criterion

If the value of an objective function (at the current point) exceeds his desirable level on this objective then he is totally satisfied with this alternative.

If, however, he value of an objective function (at the current point) is below of his reservation level on this objective then he is absolutely not satisfied with this alternative.

Consider again the bi-objective problem

$$\max\{x, 1-x\}; \text{ subject to } x \in [0,1].$$

For example, we can introduce the following linear application functions

$$h_1(t) = h_2(t) = \begin{cases} 1 & \text{if } t \geq 0.8 \\ 1 - \dfrac{0.8 - t}{0.4} & \text{if } 0.4 \leq t \leq 0.8 \\ 0 & \text{if } t \leq 0.4 \end{cases}$$

where t denotes the value attained by the objective functions.

That is,

$$h_1(f_1(x)) = \begin{cases} 1 & \text{if } f_1(x) \geq 0.8 \\ 1 - \dfrac{0.8 - f_1(x)}{0.4} & \text{if } 0.4 \leq f_1(x) \leq 0.8 \\ 0 & \text{if } f_1(x) \leq 0.4 \end{cases}$$

Introducing the notation $H_1(x) = h_1(f_1(x)$ we get

$$H_1(x) = \begin{cases} 1 & \text{if } fx \geq 0.8 \\ 1 - \dfrac{0.8 - x}{0.4} & \text{if } 0.4 \leq x \leq 0.8 \\ 0 & \text{if } x \leq 0.4 \end{cases}$$

As for the application function for the second objective function we find

$$h_2(f_2(x)) = \begin{cases} 1 & \text{if } f_2(x) \geq 0.8 \\ 1 - \dfrac{0.8 - f_2(x)}{0.4} & \text{if } 0.4 \leq f_2(x) \leq 0.8 \\ 0 & \text{if } f_2(x) \leq 0.4 \end{cases}$$

Introducing the notation

$$H_2(x) = h_2(f_2(x)) = h_2(1 - x)$$

we get

$$H_2(x) = \begin{cases} 1 & \text{if } 1 - x \geq 0.8 \\ 1 - \dfrac{0.8 - (1 - x)}{0.4} & \text{if } 0.4 \leq 1 - x \leq 0.8 \\ 0 & \text{if } 1 - x \leq 0.4 \end{cases}$$

That is,

$$H_2(x) = \begin{cases} 1 & \text{if } x \leq 0.2 \\ 1 - \dfrac{x - 0.2}{0.4} & \text{if } 0.2 \leq x \leq 0.6 \\ 0 & \text{if } x \geq 0.6 \end{cases}$$

These application functions can be interpreted as:

If one can find a feasible alternative where the values of both objectives exceeds 0.8 then the decision maker is completely satisfied with this solution.

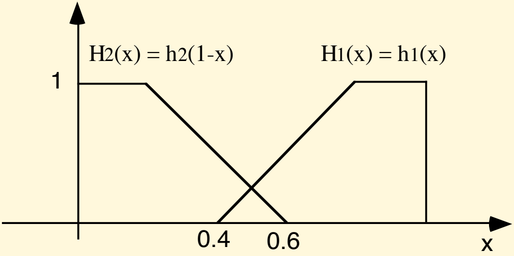

Figure 4: A simple two-objective problem.

On the other hand, alternatives for which the values of both objectives are less than 0.4 are not qualified candidates for 'good solutions'.

In other words, with the notation of

$$H_i(x) := h_i(f_i(x)),$$

the value of $H_i(x)$ may be considered as the degree of membership of x in the fuzzy set 'good solutions' for the i-th objective.

A generally used application function is the following

$$\begin{aligned} H_i(x) &= h_i(f_i(x)) \\ &= 1 - \frac{M_i - f_i(x)}{M_i - m_i}, \end{aligned}$$

where M_i denotes the independent maximum and m_i stands for the independent minimum of the i-th objective function over X.

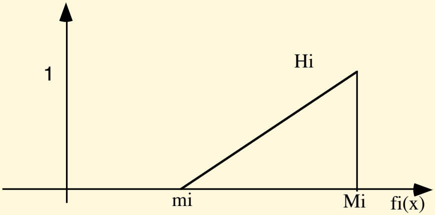

Figure 5: A simple linear application function.

It is clear that if for some alternative x^*,

$$f_i(x^*) = M_i,$$

(an ideal solution for the i-th objective) then

$$H_i(x^*) = 1,$$

since

$$\begin{aligned} H_i(x^*) &= 1 - \frac{M_i - f_i(x^*)}{M_i - m_i} \\ &= 1 - \frac{M_i - M_i}{M_i - m_i} \\ &= 1. \end{aligned}$$

and if for some alternative x_*,

$$f_i(x_*) = m_i$$

(an anti-ideal solution for the i-th objective) then $H_i(x_*)$ 0. The bigger the value of the objective the bigger the satisfaction of the decision maker.

Then a **'good compromise solution'** to MOP may be defined as an $x \in X$ being **'as good as possible'** for the whole set of objectives.

Taking into consideration the nature of H_i, it is quite reasonable to look for such a kind of solution by means of the following auxiliary problem

$$\max_{x\in X}\Big\{H_1(x),\ldots,H_k(x)\Big\}.$$

For $\max\Big\{H_1(x),\ldots,H_k(x)\Big\}$ may be interpreted as a synthetical notation of a conjunction statement **maximize jointly all objectives,** and $H_i(x)\in[0,1]$, it is reasonable to use a t-norm T to represent the connective AND. In this way

$$\max_{x\in X}\Big\{H_1(x),\ldots,H_k(x)\Big\}$$

turns into the single-objective problem

$$\max_{x\in X} T(H_1(x),\ldots,H_k(x)).$$

There exist several ways to introduce application functions.

Usually, the authors consider increasing membership functions of the form

$$h_i(t) = \begin{cases} 1 & \text{if } t \geq R_i \\ v_i(t) & \text{if } r_i \leq t \leq R_i \\ 0 & \text{if } t \leq r_i \end{cases}$$

where

$$r_i \geq m_i = \min\{f_i(x) | x \in X\}$$

denotes the **reservation level** representing minimal requirement and

$$R_i \leq M_i = \max\{f_i(x)|x \in X\}$$

denotes the **desirable level** (or reference level) on the i-th objective.

Consider again the bi-objective problem

$$\max\{x, 1-x\}; \text{ subject to } x \in [0, 1].$$

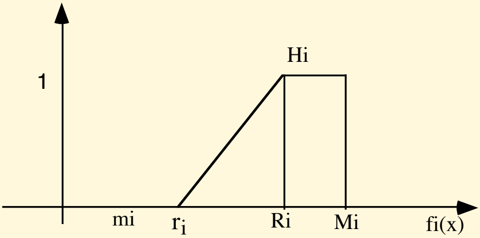

Figure 6: Linear membership function.

Using the general linear application functions

$$H_1(x) = h_1(f_1(x)) = 1 - \frac{1 - x}{1} = x,$$

$$H_2(x) = h_2(f_2(x)) = 1 - \frac{1-(1-x)}{1} = 1 - x,$$

and choosing the minimum-norm to aggregate the values of objective functions, the resulting single objective problem becomes,

$$\max \min\{x, 1-x\};\ \text{subject to } x \in [0,1].$$

has a unique solution $x^* = 1/2$ and the optimal values of the objective functions are $(0.5, 0.5)$.

If, however, we used the Łukasiewicz t-norm,

$$T_L(a,b) = \max\{a+b-1, 0\},$$

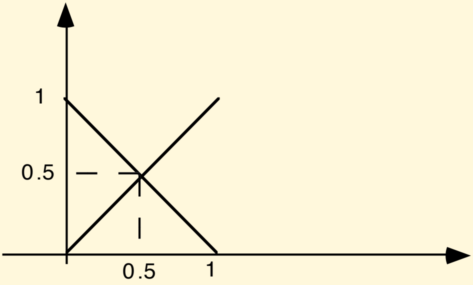

Figure 7: The optimal value is (1/2, 1/2).

for criteria aggregation then the solution set of single objective problem

$$\max \max\{x + 1 - x - 1, 0\}, \text{subject to } x \in [0, 1],$$

is $X^* = [0, 1]$.

Alternatively, we might use a non-symmetric linear application functions

$$h_1(t) = \begin{cases} 1 & \text{if } t \geq 0.8 \\ 1 - \dfrac{0.8 - t}{0.4} & \text{if } 0.4 \leq t \leq 0.8 \\ 0 & \text{if } t \leq 0.4 \end{cases}$$

$$h_2(t) = \begin{cases} 1 & \text{if } t \geq 0.8 \\ 1 - \dfrac{0.8 - t}{0.6} & \text{if } 0.2 \leq t \leq 0.8 \\ 0 & \text{if } t \leq 0.2 \end{cases}$$

where t denotes the value attained by the objective functions.

Choosing the minimum norm for aggregation, the resulting problem

$$\max \min\{H_1(x), H_2(x)\};$$
$$\text{subject to } x \in [0, 1]$$

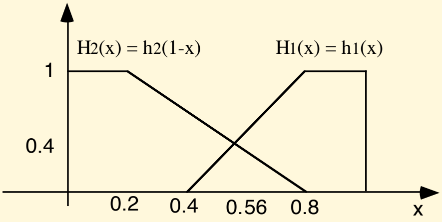

Figure 8: The optimal value is (0.4, 0.4).

has a unique solution $x^* = 0.56$ and the optimal values of the objectives are,

$$(f_1(x^*), f_2(x^*) = (0.4, 0.4).$$

3. A simple bi-objective problem

Consider the following simple bi-objective problem

$$\{x_1 + x_2, x_1 - x_2\} \to \max$$

subject to

$$0 \leq x_1, x_2 \leq 1.$$

Its Pareto optimal solutions are

$$X^* = \{(1, x_2), x_2 \in [0, 1]\}.$$

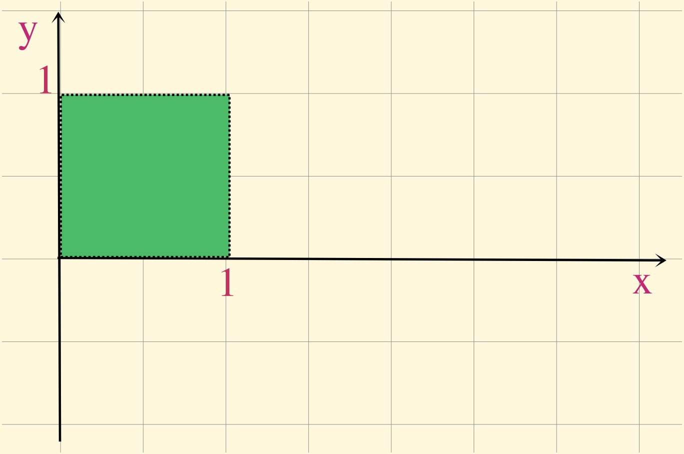

Figure 9: The decision space.

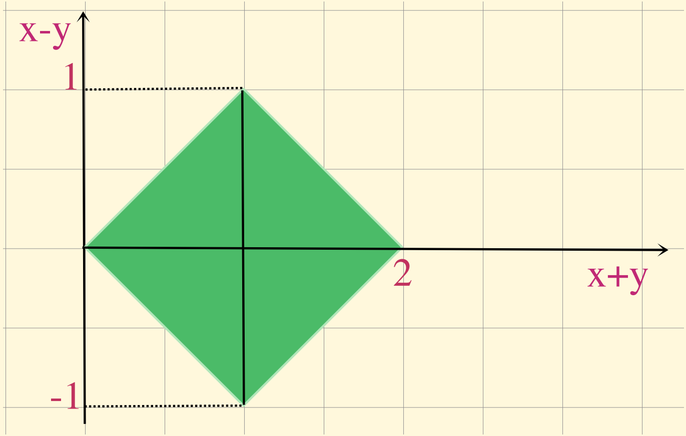

Figure 10: The criterion space.

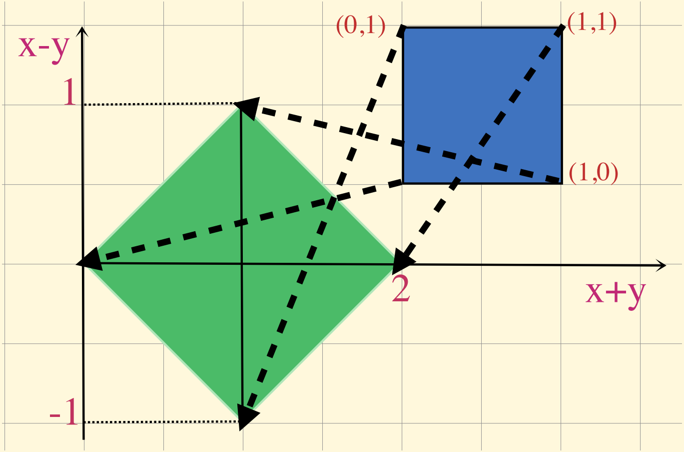

Figure 11: Explanation of the image of the decision space.

Let

$$\begin{aligned} r_1 &= m_1 \\ &= \min\{f_1(x) = x_1 + x_2 \mid 0 \le x_1, x_2 \le 1\} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} r_2 &= m_2 \\ &= \min\{f_2(x) = x_1 - x_2 \mid 0 \le x_1, x_2 \le 1\} \\ &= -1 \end{aligned}$$

be the reservation levels and let

$$
\begin{aligned}
R_1 &= M_1 \\
&= \max\{f_1(x) = x_1 + x_2 \mid 0 \le x_1, x_2 \le 1\} \\
&= 2
\end{aligned}
$$

and

$$
\begin{aligned}
R_2 &= M_2 \\
&= \max\{f_2(x) = x_1 - x_2 \mid 0 \le x_1, x_2 \le 1\} \\
&= 1
\end{aligned}
$$

be the reference points for the first and the second objectives, respectively,

$$\{x_1 + x_2, x_1 - x_2\} \to \max$$

subject to

$$0 \leq x_1, x_2 \leq 1.$$

Then we can build the following application functions

$$\begin{aligned} h_1(f_1(x)) &= h_1(x_1 + x_2) \\ &= 1 - \frac{2 - (x_1 + x_2)}{2} \\ &= \frac{x_1 + x_2}{2}, \end{aligned}$$

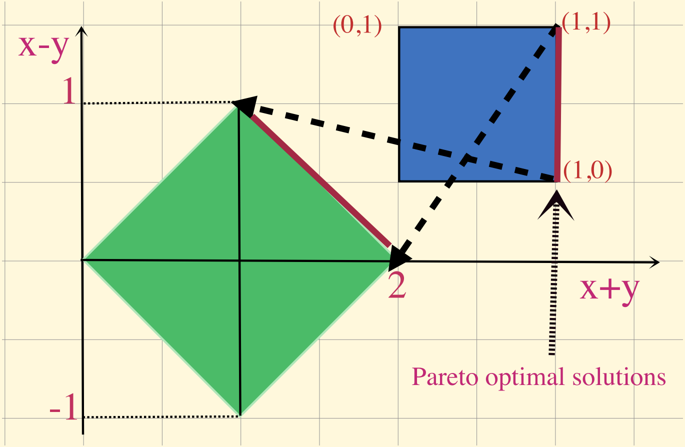

Figure 12: Pareto optimal solutions $(1, x_2)$, $x_2 \in [0, 1]$.

$$h_2(f_2(x)) = h_2(x_1 - x_2) = 1 - \frac{1 - (x_1 - x_2)}{2} = \frac{1 + x_1 - x_2}{2}.$$

Let us suppose the decision maker chooses the **minimum operator** to represent his evaluation of the connective **and** in the problem of: maximize the first objective **and** maximize the second objective.

Then the original bi-objective problem turns into the single-objective LP,

$$\max \min \left\{ \frac{x_1 + x_2}{2}, \frac{1 + x_1 - x_2}{2} \right\}$$

subject to

$$0 \leq x_1, x_2 \leq 1.$$

That is,

$$\max \lambda$$

$$\frac{x_1 + x_2}{2} \geq \lambda$$

$$\frac{1 + x_1 - x_2}{2} \geq \lambda$$

subject to

$$0 \leq x_1, x_2 \leq 1,$$

Its unique optimal solution is $x_1^* = 1$, and $x_2^* = 1$, furthermore

$$(f_1(1,1), f_2(1,1)) = (1+1, 1-1) = (2,0),$$

is a Pareto-optimal solution to the original bi-objective problem.

Suppose the decision maker chooses the Łukasiewicz t-norm, $T_L(a,b) = \max\{a+b-1, 0\}$, to represent his evaluation of the connective **and**.

Then the original bi-objective problem turns into the single-objective LP,

$$\max T_L \left\{ \frac{x_1 + x_2}{2}, \frac{1 + x_1 - x_2}{2} \right\}$$
$$\text{subject to },\ 0 \le x_1, x_2 \le 1.$$

That is,

$$\max \max\{x_1 - 1/2, 0\},$$
$$\text{subject to }\ 0 \le x_1, x_2 \le 1,$$

Its optimal solution-set is,

$$\{(1, x_2), x_2 \in [0, 1]\},$$

which is exactly, X^*, the set of Pareto optimal solutions to the original problem.

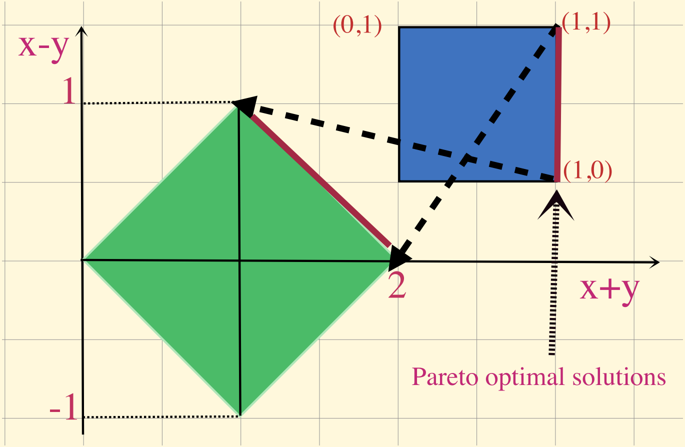

Figure 13: Pareto optimal solutions $(1, x_2)$, $x_2 \in [0, 1]$.

4. The efficiency of compromise solutions

One of the most important questions is the efficiency of the obtained compromise solutions.

Theorem 4.1. *Let* x^* *be an optimal solution to*

$$\max_{x \in X} T(H_1(x), \ldots, H_k(x))$$

where T *is a t-norm,* $H_i(x) = h_i(f_i(x))$*,* h_i *is an* increasing *application function,* $i = 1, \ldots, k$.

If h_i *is strictly increasing on the interval* $[r_i, R_i]$ *for* $i = 1, \ldots, k$*. Then* x^* *is efficient for the problem*

$$\max_{x\in X}\left\{f_1(x),\dots,f_k(x)\right\}$$

if either

(i) x^* *is unique;*

(ii) T *is strict and* $H_i(x^*) = h_i(f_i(x^*)) \in (0,1)$ *for* $i = 1,\dots,k$.

Proof. (i) Suppose that x^* is not efficient. If x^* were dominated, then $x^{**} \in X$ such that

$$f_i(x^*) \le f_i(x^{**})$$

for all i and with a strict inequality for at least one i.

Consequently, from the monotonicity of T and h_i we get

$$T(H_1(x^*), \ldots, H_k(x^*)) \leq T(H_1(x^{**}), \ldots, H_k(x^{**}))$$

which means that x^{**} is also an optimal solution to the auxiliary problem. So x^* is not unique.

(ii) Suppose that x^* is not efficient. If x^* were dominated, then $x^{**} \in X$ such that

$$f_i(x^*) \leq f_i(x^{**})$$

for all i and with a strict inequality for at least one i.

Taking into consideration that

$$H_i(x^*) = h_i(f_i(x^*)) \in (0, 1)$$

for all i and T is strict, and h_i is monoton increasing we get

$$T(H_1(x^*), \dots, H_k(x^*)) < T(H_1(x^{**}), \dots, H_k(x^{**}))$$

which means that x^* is not an optimal solution to the auxiliary problem. So x^* is not efficient. □

If we use linear application functions then they are strictly increasing on $[r_i, R_i]$, and, therefore any optimal solution x^* to the auxiliary problem is an efficient solution to the original MOP problem if either

(i) x^* is unique;

(ii) T is strict and $H_i(x^*) \in (0, 1),\ i = 1, \dots k$.

Consider the following linear bi-objective programming problem

$$\max\{2x_1 + x_2, -x_1 - 2x_2\}$$

subject to

$$x_1 + x_2 \leq 4,$$
$$3x_1 + x_2 \geq 6,$$
$$x_1, x_2 \geq 0.$$

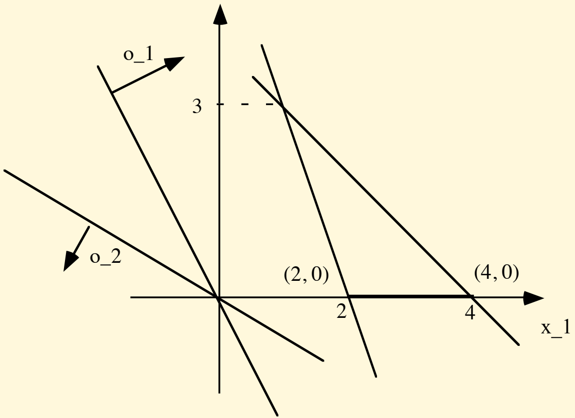

Figure 14: The bi-objective problem.

The first objective

$$2x_1 + x_2,$$

attains its maximum at point $(4, 0)$,

$$M_1 = 2x_1 + x_2 = 8,$$

whereas the second one

$$-x_1 - 2x_2,$$

has its maximum at point $(2, 0)$,

$$M_2 = -x_1 - 2x_2 = -2,$$

It is easy to see that the efficient solutions lie on the segment

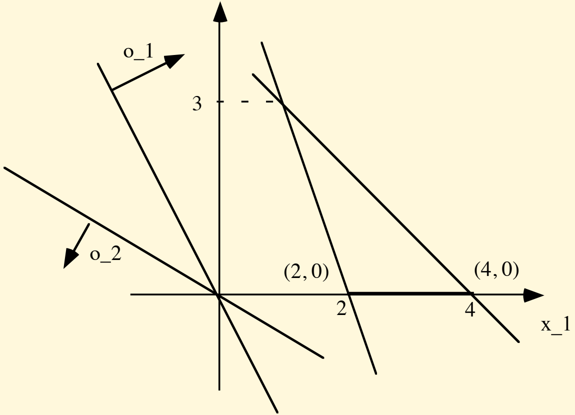

Figure 15: The bi-objective problem.

$$\{(4 - 2\gamma, 0), \gamma \in [0, 1]\}.$$

Let

$$r_1 = 4, \quad r_2 = -5$$

be the reservation levels and let

$$R_1 = 7, \quad R_2 = -3$$

be the reference points for the firts and the second objectives, respectively.

Then we can build the following application functions

$$h_1(t) = \begin{cases} 1 & \text{if } t \geq 7 \\ 1 - \dfrac{7-t}{3} & \text{if } 4 \leq t \leq 7 \\ 0 & \text{if } t \leq 4 \end{cases}$$

$$h_2(t) = \begin{cases} 1 & \text{if } t \geq -3 \\ 1 - \dfrac{-3-t}{2} & \text{if } -5 \leq t \leq -3 \\ 0 & \text{if } t \leq -5 \end{cases}$$

Let us suppose the decision maker chooses the minimum operator to represent his evaluation of the connective **and**.

Then the original bi-objective problem turns into

$$\max \min\{h_1(2x_1 + x_2), h_2(-x_1 - 2x_2)\}$$

subject to

$$x_1 + x_2 \le 4,$$
$$3x_1 + x_2 \ge 6,$$
$$x_1, x_2 \ge 0.$$

That is

$$\begin{aligned} &\max \lambda \\ \text{subject to } &h_1(2x_1 + x_2) \geq \lambda \\ &h_2(-x_1 - 2x_2) \geq \lambda \\ &x_1 + x_2 \leq 4, \\ &3x_1 + x_2 \geq 6, \\ &x_1, x_2 \geq 0. \end{aligned}$$

That is

$$\max \lambda$$

$$\text{subject to } 1 - \frac{7-(2x_1+x_2)}{3} \geq \lambda$$

$$1 - \frac{-3-(-x_1-2x_2)}{2} \geq \lambda$$

$$x_1 + x_2 \leq 4,$$

$$3x_1 + x_2 \geq 6,$$

$$x_1, x_2 \geq 0.$$

where its unique optimal solution is,

$$x^* = (23/7, 0).$$

is its optimal solution which is also efficient because it lies in the segment

$$\{(4 - 2\gamma, 0), \gamma \in [0, 1]\}.$$

The optimal values of the objective functions are

$$f_1(x^*) = 46/7$$

and

$$f_2(x^*) - 23/7.$$

This fact agrees with the theorem because x^* is the *only* optimal solution.

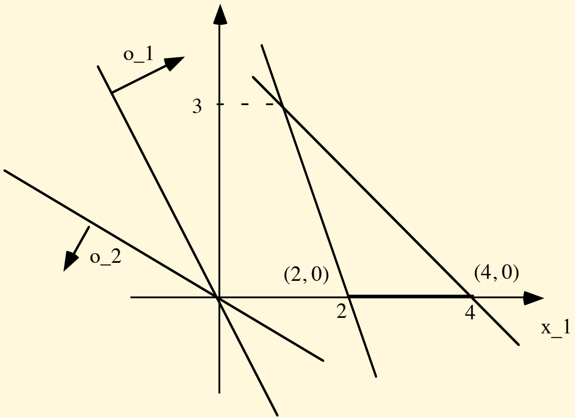

Figure 16: The bi-objective problem.