On Non-Linear Oscillation: an Analytical Series Solution

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Abstract*: In this paper, we have applied a new kind of analytical methods called''Homotopy perturbation Method" (HPM)"for two basic cases of many complecated cases.we have considered the governing equations of the problems.to show the accuracy of the method the achieved reults are compared with thoes from numerical solutions using Runge-Kutta method. It is shown that there is an excellent agreement for the whole domain. Some charts are alos presenetd to have another comparsion with energy balance method for different paprameters of the problems.*

Keywords*: non-linear oscillation; analytical solution; vibrating system; Homotopy Perturbation Method (HPM)*

1 Introduction

Differential equations are always used to represent many models of dynamical systems in physics and engineering areas. Applied mathematics is an interesting subject of engineers and scientists to prepare better understandings of the engineering problems for a long time. Finding novel techniques to apply and solve the governing differential equations has been another interesting field in mechanics and mathematics [1, 2].

The equations are complex when we have nonlinear terms in it. It is not possible to apply traditional methods to solve them or prepare an exact solution for them, but it is possible to find approximate analytical solitons to overcome the shortcomings of traditional methods and valid for whole domain of the problem. Recently, considreable attention has been paied on approximate methods such as: Homotpy Perturbation method [3-6], Energy Balance [7-13], Harmonic Balance [14], Homotopy Analysis Method [15, 16] Variational Iteration Method [17-20], Max-Min [21-25], Differential Transform [26], Amplitude Frequency Formulation [27-28] and Adomian Decomposition [29], Hamiltonian approach [30],Variational approach [31].

This paper considers the following general nonlinear oscillators:

$$u''+\omega_0^2 u+\varepsilon f(u)=0 \tag{1}$$

With initial conditions:

$$u(0)=A\ ,\ u'(0)=0 \tag{2}$$

where f is a nonlinear function of u'', u', u in this preliminary report, we suppose the simplest case, i.e., f depends upon only the function of u. If there is no small parameter in the equation, traditional perturbation methods cannot be applied directly. Recently, considerable attention has been paid to the analytical solutions for nonlinear equations without possible small parameters. Traditional perturbation methods have many shortcomings, and they are not valid for strongly nonlinear equations. Mechanical oscillatory systems are often governed by nonlinear differential equations. It is well known that a nonlinear equation of this type is often linearized by retaining the first term of the Taylor series expansion of the restoring force in a neighborhood of the equilibrium point. This procedure yields acceptable results for many cases, but is unable to show the amplitude dependence of the oscillation period.

2 Overview of the Analytical Method

To explain the basic idea of the HPM for solving nonlinear differential equations we consider the following nonlinear differential equation:

$$A(u)-f\ (r)=0,\ \ r\in\Omega \tag{3}$$

Subject to boundary condition

$$B(u,\partial u/\partial n)=0,\ \ r\in\Gamma \tag{4}$$

where A is a general differential operator, B a boundary operator, $f(r)$ is a known analytical function, Γ is the boundary of domain Ω and $\partial u/\partial n$ denotes differentiation along the normal drawn outwards from Ω .The operator A can, generally speaking, be divided into two parts: a linear part L and a nonlinear part N. equation (3) therefore can be rewritten as follows:

$$L(u)+N(u)-f(r)=0 \tag{5}$$

In case that the nonlinear equation (3) has no "small parameter", we can construct the following Homotopy:

$$H(v,p)=L(v)-L(u_0)+pL(u_0)+p(N(v)-f(r))=0 \tag{6}$$

Where,

$$v(r,p):\ \Omega\times[0,1]\rightarrow R \tag{7}$$

In equation (9), $p\in[0,1]$ is an embedding parameter and u_0 is the first approximation that satisfies the boundary condition. We can assume that the solution of equation (6) can be written as a power series in p, as following:

$$v=v_0+pv_1+p^2v_2+..., \tag{8}$$

And the best approximation for solution is:

$$u=\lim_{p\rightarrow 1}v=v_0+v_1+v_2+..., \tag{9}$$

When, Eq. (6) correspond to equation (3) and equation (9) becomes the approximate solution of equation (3).

3 Description of the First Case

In this section we consider a rigid frame (Fig. 1) which is forced to rotate at the fixed rate Ω. While the frame rotates, the simple pendulum oscillates.

The governing equation is:

$$\ddot{\theta}+(1-\beta\cos\theta)\sin\theta=0\ ,\ \theta(0)=A\ ,\ \ddot{\theta}(0)=0 \tag{10}$$

Where

$$\beta=\frac{\Omega^2 r}{g}<1$$

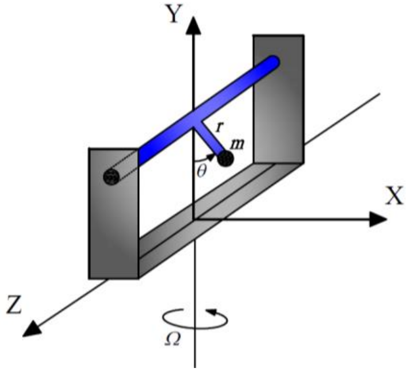

Figure 1
Rotating frame connected to a pendulum

3.1 Application of Homotopy Perturbation Method

We use Taylor series instead of functions of sin (θ(t)) and cos (θ(t)):

$$\sin(\theta) = \theta - \frac{1}{6}\theta^3 \tag{11}$$

$$\cos(\theta) = 1 - \frac{1}{2}\theta^2 \tag{12}$$

We substitute equations (11), (12) in equation (10), so:

$$\ddot{\theta} + \left(1 - \beta\left(1 - \frac{\theta^2}{2}\right)\right)\left(\theta - \frac{\theta^3}{6}\right) \tag{13}$$

Separate the linear and nonlinear part:

$$L : \ddot{\theta} + \theta - \beta\theta \tag{14}$$

$$N : \ddot{\theta} + \theta - \beta\theta - \frac{1}{6}\theta^3 + \frac{2}{3}\beta\theta^3 - \frac{1}{12}\beta\theta^5 \tag{15}$$

Now we form the Homotopy as follows:

$$H(\theta, p) = (1-p)\left(\ddot{\theta} + \theta - \beta\theta\right) + p\left(\ddot{\theta} + \theta - \beta\theta - \frac{\theta^3}{6} + \frac{2}{3}\beta\theta^3 - \frac{1}{12}\beta\theta^5\right) \tag{16}$$

$$\theta(t) = \theta_0 + p\theta_1 \tag{17}$$

By substituting equation (17) in equation (16):

$$\begin{aligned} &H(\theta,p)=(1-p)\left[\ddot{\theta}_0+p\ddot{\theta}_1+\theta_0+p\theta_1-\beta(\theta_0+p\theta_1)\right] \\ &+p[\ddot{\theta}_0+p\ddot{\theta}_1+\theta_0+p\theta_1-\frac{1}{6}(\theta_0+p\theta_1)^3-\beta(\theta_0+p\theta_1) \\ &\quad+\frac{2}{3}\beta(\theta_0+p\theta_1)^3-\frac{1}{12}\beta(\theta_0+p\theta)_1^5] \end{aligned} \tag{18}$$

We simplified equation (18) on the basis of p powers:

$$\begin{aligned} &H(\theta,p)=-\frac{1}{12}\beta\theta_1^5p^6-\frac{5}{12}\beta\theta_0\theta_1^4p^5+ \\ &\left[-\frac{1}{6}\theta_1^3+\frac{2}{3}\beta\theta_1^3-\frac{5}{6}\beta\theta_0{}^2\theta_1^3\right]p^4+\left[2\beta\theta_0\theta_1^2-\frac{5}{6}\beta\theta_0{}^3\theta_1^2-\frac{1}{2}\theta_0\theta_1^2\right]p^3+ \\ &\left[-\frac{1}{2}\theta_0{}^2\theta_1-\frac{5}{12}\beta\theta_0{}^4\theta_1+2\beta\theta_0{}^2\theta_1\right]p^2+\left[\theta_1-\frac{1}{12}\beta\theta_0{}^5+\frac{2}{3}\beta\theta_0{}^3-\frac{1}{6}\theta_0{}^3+\theta_1''-\beta\theta_1\right]p+ \\ &\theta_0+\theta_0''-\beta\theta_0 \end{aligned} \tag{19}$$

Now we should solve these equations:

$$p^0:\ \ddot{\theta}_0+\theta_0-\beta\theta_0 \quad \theta_0(0)=A\ ,\ \ddot{\theta}_0(0)=0 \tag{20}$$

So:

$$\theta_0(t)=A\cos(\sqrt{1-\beta t}) \tag{21}$$

And

$$\begin{aligned} &p^1:\ddot{\theta}_1+\theta_1-\beta\theta_1-\frac{1}{12}\beta A^5\cos(\sqrt{1-\beta t})^5+ \\ &\quad\frac{2}{3}\beta A^3\cos(\sqrt{1-\beta t})^3-\frac{1}{6}A^3\cos(\sqrt{1-\beta t})^3 \qquad \theta_1(0)=0\ ,\ \dot{\theta}_1(0)=0 \end{aligned} \tag{22}$$

So:

$$\begin{aligned} &\theta_1(t)=\frac{1}{72}\frac{\cos\left(\sqrt{1-\beta t}\right)\left(\beta A^2-12\beta+3\right)A^3}{\beta-1} \\ &-\frac{1}{4608\beta-4608}\left[120A^3\left(\left(-\frac{1}{8}\beta A^2+\frac{4}{5}\beta-\frac{1}{5}\right)\cos\left(3\sqrt{1-\beta t}\right)-\frac{1}{120}\beta A^2\cos\left(5\sqrt{1-\beta}\,t\right)+\right.\right. \\ &\left.\left.\left(\frac{2}{3}\beta A^2-\frac{36}{5}\beta+\frac{9}{5}\right)\cos\left(\sqrt{1-\beta t}\right)+\sqrt{1-\beta}\left(\beta A^2-\frac{48}{5}+\beta\frac{12}{5}\right)\sin\left(\sqrt{1-\beta t}\right)t\right)\right) \end{aligned} \tag{23}$$

As seen before in equation (17) we have:

$$\begin{aligned} &\theta_{HPM}=A\cos(\sqrt{1-\beta t})+\frac{1}{72}\frac{\cos\left(\sqrt{1-\beta t}\right)\left(\beta A^2-12\beta+3\right)A^3}{\beta-1} \\ &-\frac{1}{4608\beta-4608}\left[120A^3\left(\left(-\frac{1}{8}\beta A^2+\frac{4}{5}\beta-\frac{1}{5}\right)\cos\left(3\sqrt{1-\beta t}\right)-\frac{1}{120}\beta A^2\cos\left(5\sqrt{1-\beta}\,t\right)+\right.\right. \\ &\left.\left.\left(\frac{2}{3}\beta A^2-\frac{36}{5}\beta+\frac{9}{5}\right)\cos\left(\sqrt{1-\beta t}\right)+\sqrt{1-\beta}\left(\beta A^2-\frac{48}{5}+\beta\frac{12}{5}\right)\sin\left(\sqrt{1-\beta t}\right)t\right)\right) \end{aligned} \tag{24}$$

3.2 Results and Discussions

To show the accuracy of the results, the HPM solutions are compared with the numerical ones. Table 1 represents the comparsion of HPM and Runge-Kutta for different values.

A brief discussion of the numerical method used in this paper is described in Appendix A. Then, we have compared the results with those obtained by EBM solution, in Figs. 2 and 3:

$$\omega_{EBM} = A\cos\left(\frac{2}{A}\sqrt{\cos\left(\frac{\sqrt{2}}{2}A\right)-\cos A+\frac{\beta}{2}\left(\cos^2 A-\cos^2\left(\frac{\sqrt{2}}{2}A\right)\right)}\times t\right) \tag{25}$$

As it is evident, a same manner with a high accuracy is gained by HPM. The EBM is shortly explained in Appendix B.

For table 1

Case 1: $g=10,\ r=2,\ \Omega=1,\ A=\pi/12$

Case 2: $g=10,\ r=0.5,\ \Omega=2.5,\ A=\pi/4$

Table 1
The results of HPM and numerical solution for $\theta(t)$

	Case 1			Case 2		
T	**HPM**	**Numerical solution**	**Error**	**HPM**	**Numerical solution**	**Error**
0	0.2618	0.2618	0.0000	0.7854	0.7854	0.0000
0.4	0.2428	0.2453	0.0099	0.7448	0.7416	0.0043
0.8	0.1958	0.1978	0.0099	0.6180	0.6153	0.0043
1.2	0.1240	0.1252	0.0099	0.4225	0.4207	0.0043
1.6	0.0365	0.0369	0.0098	0.1806	0.1798	0.0042
2	-0.0556	-0.0562	0.0100	-0.0811	-0.0807	0.0049
2.4	-0.1407	-0.1421	0.0100	-0.3340	-0.3325	0.0045
2.8	-0.2079	-0.2100	0.0099	-0.5502	-0.5477	0.0045
3.2	-0.2489	-0.2514	0.0099	-0.7056	-0.7024	0.0046
3.6	-0.2585	-0.2611	0.0099	-0.7826	-0.7789	0.0047
4	-0.2354	-0.2377	0.0099	-0.7724	-0.7687	0.0048
4.4	-0.1826	-0.1844	0.0098	-0.6761	-0.6728	0.0050
4.8	-0.1067	-0.1077	0.0098	-0.5047	-0.5021	0.0050
5.2	-0.0172	-0.0174	0.0092	-0.2774	-0.2761	0.0050
5.6	0.0744	0.0751	0.0101	-0.0197	-0.0197	0.0004
6	0.1565	0.1581	0.0100	0.2401	0.2388	0.0057
6.4	0.2189	0.2211	0.0099	0.4737	0.4711	0.0055
6.8	0.2536	0.2562	0.0099	0.6552	0.6515	0.0056

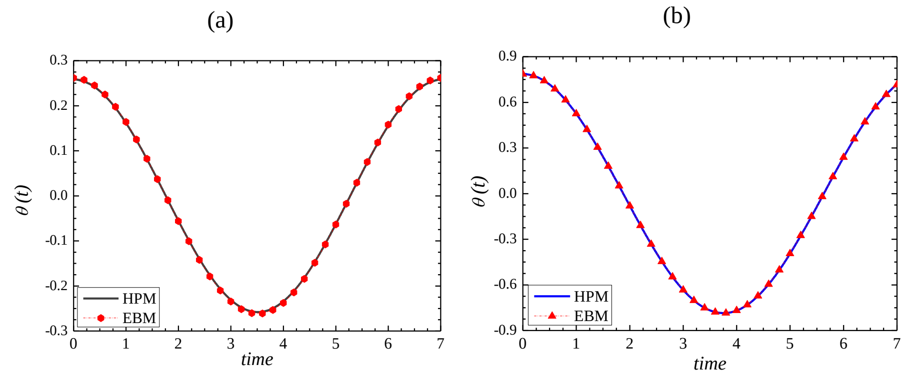

Figure 2

Comparison of time history response of HPM solution with the EBM solution for

(a) $g=10,\ r=2,\ \Omega=1,\ A=\pi/12$, (b) $g=10,\ r=0.5,\ \Omega=2.5,\ A=\pi/4$

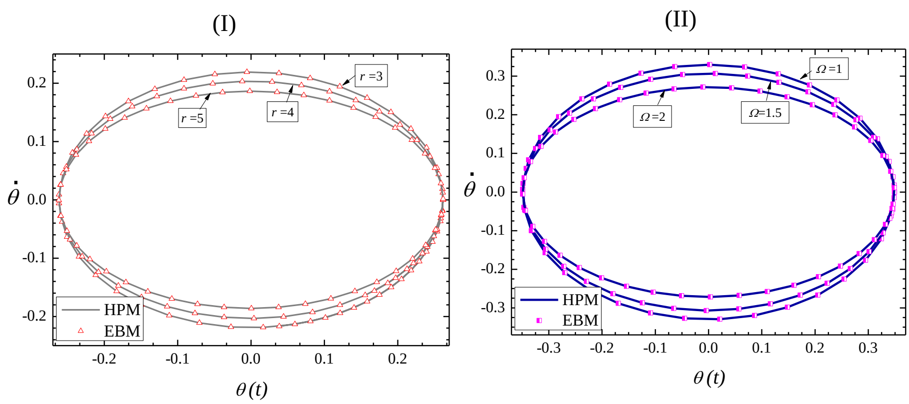

Figure 3

Comparison of phase plan of HPM solution with EBM solution for variation parameter

(I) $g=10,\ r=2,\ A=\pi/12$, (II) $g=10,\ \Omega=1,\ A=\pi/9$

4 Description of the Second Problem

We first consider the following nonlinear oscillator (Fig. 4):

$$\left(1+Ru^2\right)\ddot{u}+Ru\dot{u}^2+\omega_0^2u+\frac{1}{2}\frac{Rgu^3}{l}=0 \qquad u(0)=A,\dot{u}(0)=0 \tag{26}$$

where

$$\omega_0^{\,2} = \frac{k}{m_1} + \frac{Rg}{l} \;, \quad R = \frac{m_2}{m_1} \;, \quad \left| u = \frac{x}{l} \right| << 1 \tag{27}$$

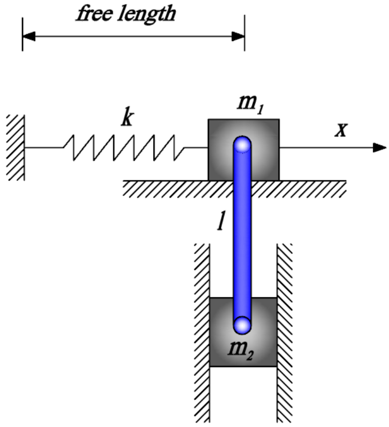

Figure 4
Geometry of Example 2

4.1 Implementation of Homotopy Perturbation Method

The governing equation is:

$$\left(1+Ru^2\right)\ddot{u} + Ru\dot{u}^2 + \omega_0^2 u + \frac{1}{2}\frac{R\,g\,u^3}{l} \tag{28}$$

Separate the linear and nonlinear part:

$$L : \ddot{u} + \omega_0^2 u \tag{29}$$

$$N : \left(1+Ru^2\right)\ddot{u} + Ru\dot{u}^2 + \omega_0^2 u + \frac{1}{2}\frac{R\,g\,u^3}{l} \tag{30}$$

Then we create the Homotopy Perturbation:

$$\begin{aligned} H(u,p) &= (1-p)\left[\ddot{u} + \omega_0^2 u\right] \\ &\quad + p\left[\left(1+Ru^2\right)\ddot{u} + Ru\dot{u}^2 + \omega_0^2 u + \frac{1}{2}\frac{R\,g\,u^3}{l}\right] \end{aligned} \tag{31}$$

$$u(t) = u_0 + pu_1 \tag{32}$$

Substitute equation (32) in equation (31), we have:

$$\begin{aligned}H(u,p) &= (1-p)\left[\ddot{u}_0 + p\ddot{u}_1 + \omega_0^2(u_0+pu_1)\right] + \\ &p\Big[\left(1+R(u_0+pu_1)^2\right)(\ddot{u}_0+p\ddot{u}_1) + R(u_0+pu_1)(\dot{u}_0+p\dot{u}_1)^2 \\ &\quad + \omega_0^2(u_0+pu_1) + \frac{1}{2}\frac{R\,g\,(u_0+pu_1)^3}{l}\Big]\end{aligned} \tag{33}$$

We simplified equation (33) on the basis of p powers:

$$\begin{aligned}H(u,p) &= \left[\frac{1}{2}\frac{Rgu_1^3}{l} + Ru_1^2\ddot{u}_1 + Ru_1\dot{u}_1^2\right]p^4 \\ &+ \left[Ru_1^2\ddot{u}_0 + Ru_0\dot{u}_1^2 + \frac{3}{2}\frac{R\,g\,u_0u_1^2}{l} + 2Ru_1\dot{u}_1\dot{u}_0 + 2Ru_0u_1\ddot{u}_1\right]p^3 \\ &+ \left[Ru_0^2\ddot{u}_1 + \frac{3}{2}\frac{Rg\,u_0^2u_1}{l} + Ru_1\dot{u}_0^2 + 2Ru_0\dot{u}_0\dot{u}_1 + 2Ru_0u_1\ddot{u}_0\right]p^2 \\ &+ \left[\ddot{u}_1 + Ru_0\dot{u}_0^2 + Ru_0^2\ddot{u}_0 + \omega_0^2u_1 + \frac{1}{2}\frac{Rgu_0^3}{l}\right]p + \ddot{u}_0 + \omega_0^2u_0\end{aligned} \tag{34}$$

Now we should solve these equations:

$$p^0 : \ddot{u}_0 + \omega_0^2u_0 \qquad u_0(0) = A\ ,\ \dot{u}_0(0) = 0 \tag{35}$$

So:

$$u_0(t) = A\cos(\omega_0 t) \tag{36}$$

And

$$p^1 : \ddot{u}_1 + Ru_0\dot{u}_0^2 + Ru_0^2\ddot{u}_0 + \omega_0^2u_1 + \frac{1}{2}\frac{Rgu_0^3}{l} \qquad u_1(0) = 0\ ,\ \dot{u}_1(0) = 0 \tag{37}$$

We substitute equation (36) in equation (37) and solve it with the given boundary conditions:

$$\begin{aligned}u_1(t) &= -\frac{1}{64}\frac{A^3Rg\cos(\omega_0 t)}{\omega_0^2 l} + \frac{1}{4}A^3R\cos(\omega_0 t) + \frac{1}{64}A^3Rg\cos(3\omega_0 t) \\ &- \frac{3}{16}\frac{A^3Rg\,\omega_0\sin(\omega_0 t)t}{\omega_0^2 l} - \frac{1}{16}A^3R\cos(3\omega_0 t) + \frac{1}{4}A^3R\sin(\omega_0 t)t\end{aligned} \tag{38}$$

As seen before in equation (32) we have:

$$\begin{aligned}u_{HPM} &= A\cos(\omega_0 t) - \frac{1}{64}\frac{A^3Rg\cos(\omega_0 t)}{\omega_0^2 l} + \frac{1}{4}A^3R\cos(\omega_0 t) + \frac{1}{64}A^3Rg\cos(3\omega_0 t) \\ &- \frac{3}{16}\frac{A^3Rg\,\omega_0\sin(\omega_0 t)t}{\omega_0^2 l} - \frac{1}{16}A^3R\cos(3\omega_0 t) + \frac{1}{4}A^3R\sin(\omega_0 t)t\end{aligned} \tag{39}$$

4.2 Results and Ddiscussions

In this part, the results are compared with numerical solutions. The accuracy of the method is shown in Table 2. Figs. 5 and 6 represent the comparsion of the HPM and EBM results.

$$u_{EBM} = A\cos\left(\sqrt{\frac{\frac{1}{4}\omega_0^2 A^2 + \frac{3Rg}{32l}A^4}{\frac{1}{4}A^2 + \frac{1}{8}A^4}} \times t\right) \tag{40}$$

For Table 2

Case1: $g = 10, m_1 = 4, m_2 = 2, l = 1, k = 20, \tan(\pi/6)$

Table 2

The results of HPM and numerical solution for *u(t)*

	Case 1			Case 2		
T	**HPM**	**Numerical solution**	**Error**	**HPM**	**Numerical solution**	**Error**
0	0.5774	0.5774	0.0000	0.8038	0.803848	0.0000
0.05	0.5155	0.5167	0.0024	0.6785	0.677134	0.0020
0.1	0.3376	0.3396	0.0061	0.3268	0.325309	0.0047
0.15	0.0745	0.0758	0.0168	-0.1450	-0.14584	0.0059
0.2	-0.2089	-0.2079	0.0050	-0.5614	-0.56223	0.0014
0.25	-0.4368	-0.4360	0.0019	-0.7859	-0.7841	0.0023
0.3	-0.5607	-0.5626	0.0034	-0.7646	-0.75656	0.0106
0.35	-0.5628	-0.5707	0.0138	-0.4937	-0.48546	0.0170
0.4	-0.4429	-0.4535	0.0235	-0.0452	-0.04258	0.0608
0.45	-0.2179	-0.2243	0.0288	0.4171	0.416369	0.0018
0.5	0.0648	0.0632	0.0260	0.7325	0.725416	0.0097
0.55	0.3298	0.3311	0.0038	0.8168	0.797864	0.0238
0.6	0.5113	0.5188	0.0145	0.6395	0.618087	0.0346
0.65	0.5773	0.5960	0.0314	0.2365	0.228363	0.0356
0.7	0.5196	0.5452	0.0471	-0.2504	-0.24637	0.0163
0.75	0.3452	0.3638	0.0512	-0.6450	-0.6296	0.0245
0.8	0.0842	0.0888	0.0520	-0.8335	-0.79984	0.0421
0.85	-0.1999	-0.2055	0.0276	-0.7588	-0.71746	0.0577
0.9	-0.4306	-0.4471	0.0369	-0.4218	-0.40031	0.0537
0.95	-0.5585	-0.5917	0.0562	0.0668	0.061481	0.0861
1	-0.5648	-0.6113	0.0762	0.5260	0.500082	0.0519

Case 2: $g = 10, m_1 = 5, m_2 = 1, l = 1, k = 50, A = 3\tan(\pi/12)$

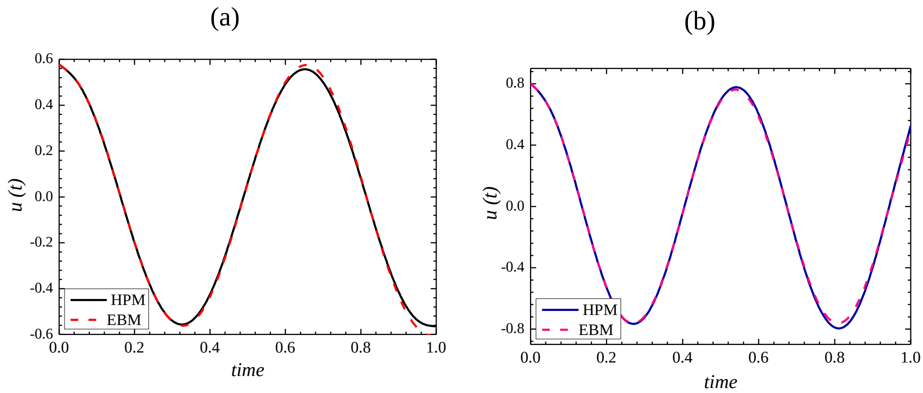

Figure 5

Comparison of time history response of HPM solution with the EBM solution for

(a) $g=10, m_1=4, m_2=2, l=1, k=20, \tan(\pi/6)$,

(b) $g=10, m_1=5, m_2=1, l=1, k=50, A=3\tan(\pi/12)$

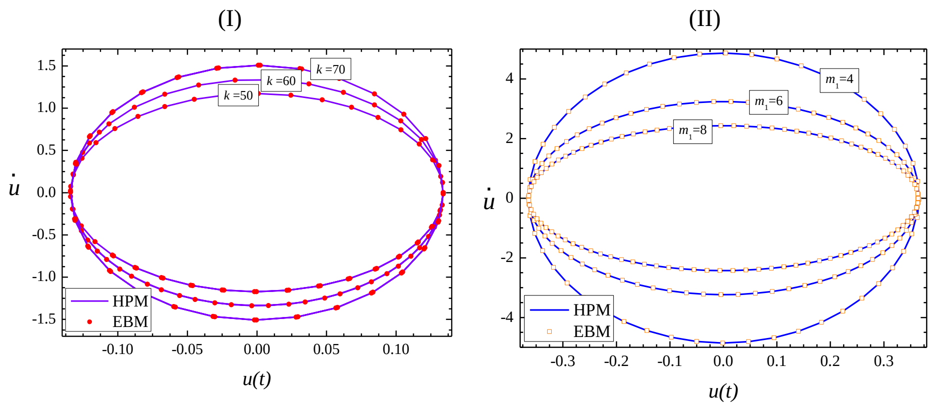

Figure 6

Comparison of phase plan of HPM solution with EBM solution for variation parameter

(I): $g=10, m_1=8, m_2=2, l=1, A=0.5\tan(\pi/12)$,

(II): $g=10, m_2=2, l=1.5, k=40, A=\tan(\pi/9)$

Conclusion

In this paper Homotopy Perturbation Method was used to solve some practical non-linear equation of oscillators. It was observed that the results obtained by HPM are in very good agreement with those achieved by numerical solution and another analytical technique namely EBM. It is an advantageous to solve oscillation systems as such problems frequently arise in many branches of sciences and engineering while HPM is much simpler in comparison with other methods. Besides, the exact expression obtained here can be used in a wide range of future numerical or analytical investigations. Following the successful application of the method introduced here for highly non-linear oscillators, HPM is strongly proposed by the authors in exploring solutions to the same problems.

Appendix A. DESCRIPTION OF RK4

The Runge-Kutta method is an important iterative method for the approximation solutions of ordinary differential equations. These methods were developed by the German mathematician Runge and Kutta around 1900. For simplicity, we explain one of the important methods of Runge-Kutta methods, called forth-order Runge-Kutta method.

Consider an initial value problem be specified as follows:

$$y' = f(t, y), \quad y(t_0) = y_0 \tag{A. 1}$$

Then RK4 method is given for this problem as below:

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4), \\ t_{n+1} &= t_n + h. \end{aligned} \tag{A. 2}$$

Where y_{n+1} is the RK4 approximation of $y(t_{n+1})$ and

$$\begin{aligned} k_1 &= f\left(t_n, y_n\right), \\ k_2 &= f\left(t_n + \frac{1}{2}h,\, y_n + \frac{1}{2}hk_1\right), \\ k_3 &= f\left(t_n + \frac{1}{2}h,\, y_n + \frac{1}{2}hk_3\right), \\ k_3 &= f\left(t_n + h,\, y_n + hk_3\right). \end{aligned} \tag{A. 3}$$

Appendix B. An Introduction to Energy Balanc Method

Consider a general nonlinear oscillator in the form:

$$u'' + f(u(t)) = 0 \tag{B.1}$$

in which u and t are generalized dimensionless displacement and time variables, respectively. Its variational principle can be easily obtained:

$$J(u) = \int_0^t (-\frac{1}{2}u'^2 + F(u))dt \tag{B.2}$$

Where $T = \frac{2\pi}{\omega}$ is period of the nonlinear oscillator, $F(u) = \int f(u)du$.

Its Hamiltonian, therefore, can be written in the form:

$$H = \frac{1}{2}u'^2 + F(u) + F(A) \tag{B.3}$$

$$R(t) = \frac{1}{2}u'^2 + F(u) - F(A) = 0 \tag{B.4}$$

Oscillatory systems contain two important physical parameters, i.e., the frequency ω and the amplitude of oscillation, *A*. So let us consider such initial conditions:

$$u(0) = 0 \ , \ u'(0) = 0 \tag{B.5}$$

We use the following trial function to determine the angular frequency ω:

$$u(t) = A\cos(\omega t) \tag{B.6}$$

Substituting (B.6) into *u* term of (B.4), yield:

$$R(t) = \frac{1}{2}\omega^2 A^2 \sin^2 \omega t + F(A\cos\omega t) - F(A) = 0 \tag{B.7}$$

If, by chance, the exact solution had been chosen as the trial function, then it would be possible to make R zero for all values of t by appropriate choice of ω. Collocation at $\omega t = \frac{\pi}{4}$ gives:

$$\omega = \sqrt{\frac{2(F(A) - F(A\cos\omega t)}{A^2 \sin^2 \omega t}} \tag{B.8}$$

$$T = \frac{2\pi}{\sqrt{\frac{2(F(A) - F(A\cos\omega t))}{A^2 \sin^2 \omega t}}} \tag{B.9}$$

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