On the Asymptotic Behavior of Solutions of Neurodynamical Systems

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Abstract: In the present paper we investigate the dynamic properties of a specific class of nonlinear delay-differential equations by studying the asymptotic behavior of their solutions by means of Lyapunov's exponent. Systems of delay-differential equations can be used to model recurrent neural networks.

Keywords: Neural networks, Lyapunov's exponent, Cauchy´s matrix, Neurodynamical system.

1 Introduction

The stability of nonlinear dynamical system is a difficult issue to deal with. When we speak of stability in the context of nonlinear dynamical system, we usually mean stability in the sense of Lyapunov. A. M. Lyapunov (see [10]) presented the fundamental concepts of the stability theory known as the first method of Lyapunov. This method is widely used for the stability analysis of linear and nonlinear systems, both time-invariant and time-varying. As such it is directly applicable to the stability analysis of neural networks. The study of neurodynamics may follow one of two routes, depending on the application of interest:

1 Deterministic neurodynamics, in which the neural network model has a deterministic behavior. In mathematical terms, it is described by a set of nonlinear delay-differential equations that define the exact evolution of the model as a function of time.

2 Statistical neurodynamics, in which the neural network model is perturbed by the presence of noise. In this case, we have to deal with stochastic nonlinear differential equations, expressing the solution in probabilistic terms. The combination of stochastic and nonlinearity makes the subject more difficult to handle.

In this paper we restrict ourselves to deterministic neurodynamics.

2 Definitions of Lyapunov´s Exponent

In order to proceed with the study of neurodynamics, we need a mathematical model for describing the dynamics of nonlinear system. A model most naturally suited for this purpose is the so-called state-space model. According to this model, we think in terms of a set of state variables whose values are supposed to contain sufficient information to predict the future evolution of the system. Let $x_1(t), x_2(t), \ldots, x_n(t)$ denote the state variables of a nonlinear dynamical system, where continuous time *t* is the independent variable and *n* is the order of system. The dynamics of a large class of nonlinear dynamical systems may then be cast in the form of a system of first-order delay-differential equations written as follows:

$$\frac{d}{dt}x_i(t) = \sum_{j=1}^{n} p_{ij}(t)x_j(t) + \sum_{j=1}^{n} q_{ij}(t)u_j(t) + I_i(t) \qquad (1)$$

where all functions $p_{ij}(t), q_{ij}(t), I_i(t)$ are assumed to be continuous functions of time,

$p_{ii}(t) < 0$, $\quad |p_{ij}(t)| \le p, \quad |q_{ij}(t)| \le q,\ u_j(t) = \frac{1}{2}x_j(t)\left(|x_j(t-\tau)+1| - |x_j(t-\tau)-1|\right)$,

$\tau > 0$, $t-\tau < t_0, \quad i = 1, 2, \ldots, n$.

This system of delay – differential equations can be used to model neural networks.

The initial value problem (IVP) for (1) is defined as follows:

On the initial set $E_{t_0} = \{t-\tau : t-\tau < t_0, t \in \langle t_0, \infty)\} \cup \{t_0\}$

let a continuous initial vector functions $\varphi(t) = (\varphi_0(t), \varphi_1(t), \varphi_2(t), \ldots, \varphi_{n-1}(t))$

be given.

We have to find the solution $x(t) \in C^n(\langle t_0, \infty))$ of (1) $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$, satisfying

$$x_{j+1}(t) = \varphi_j(t) \text{ if } t-\tau \le t \le t_0,\ j = 0, 1, 2, \ldots, n-1. \qquad (2)$$

Under the above assumptions, the initial value problem (1), (2) has exactly one solution on the interval $\langle t_0, \infty)$ where

$$\varphi_j(t) = x_{j+10}\psi_j(t), \quad x_{j+1}(t_0) = x_{j+10}, \quad \psi_j(t_0) = 1, \quad j = 0, 1, \dots, n-1 \cdot$$

In the following we consider the system of nonlinear delay-differential equations of the form

$$\frac{d}{dt} x_i(t) = \sum_{j=1}^{n} p_{ij}(t) x_j(t) + \sum_{j=1}^{n} q_{ij}(t) u_j(t), \quad i = 1, 2, \dots, n. \tag{3}$$

Definition 1.1 A superior Lyapunov's exponent of a vector function $x(t)$ is called a real number $\overline{\lambda}$ which is defined by

$$\overline{\lambda} = \limsup_{t\to\infty}\left(\frac{1}{t}\ln\|x(t)\|\right) \cdot$$

Definition 1.2 A inferior Liapunov's exponent of a vector function $x(t)$ is called a real number $\underline{\lambda}$ which is defined by

$$\underline{\lambda} = \liminf_{t\to\infty}\left(\frac{1}{t}\ln\|x(t)\|\right)$$

where

$$\|x\| = \sqrt{(x, x)} \quad \text{and} \quad (x, y) = \sum_{i=1}^{n} x_i \overline{y}_i \cdot$$

Definition 1.3 A superior central exponent of Cauchy's matrix of a linear differential system is called a real number Ω which is defined by

$$\Omega = \inf_{T>0}\left(\limsup_{k\to\infty}\frac{1}{kT}\sum_{i=1}^{k}\ln\|X(iT,(i-1)T)\|\right) = \lim_{T\to+\infty}\left(\limsup_{k\to\infty}\frac{1}{kT}\sum_{i=1}^{k}\ln\|X(iT,(i-1)T)\|\right).$$

Definition 1.4 A inferior central exponent of Cauchy's matrix of a linear differential system is called a real number ω which is defined by

$$\omega = \inf_{T>0}\left(\limsup_{k\to\infty}\frac{1}{kT}\sum_{i=1}^{k}\ln\|X^{-1}(iT,(i-1)T)\|^{-1}\right) =$$

$$= \lim_{T\to+\infty}\left(\limsup_{k\to\infty}\frac{1}{kT}\sum_{i=1}^{k}\ln\|X^{-1}(iT,(i-1)T)\|^{-1}\right).$$

We have to find the norm of Cauchy's matrix of the linear differential system by using the following formula

$$\|X(t,s)\| = \max_{x}\frac{\|x(t)\|}{\|x(s)\|}$$

where we have to search a maximum element of a set of all solutions of the linear differential system.

Choose any nontrivial solution $w(t)=\left(w_1(t), w_2(t), \dots, w_n(t)\right)$ of the set of all solutions of (3).

If $a_{w,ij}(t)$ denotes $p_{ij}(t)+q_{ij}(t)v_j(t)$, $\quad í=1,2,\dots,n, \quad j=1,2,\dots n$

and $v_j(t)$ denotes $\frac{1}{2}\left(\left|w_j(t-\tau)+1\right|-\left|w_j(t-\tau)-1\right|\right)$,

then $w(t)$ is the solution of the linear differential system

$$\frac{d}{dt}y_i(t)=\sum_{j=1}^{n}a_{w,ij}(t)y_j(t), \quad i=1,2,\dots,n \tag{4}$$

too. The equality

$$a_{w,ij}(t)= p_{ij}(t)+ q_{ij}(t)v_j(t) \tag{5}$$

implies the fact that all coefficients $a_{w,ij}(t)$ are continuous functions of time and

$$\begin{aligned}&\left|v_j(t)\right|\le 1, \quad t\in\langle 0,+\infty), \quad \left|a_{w,ij}(t)\right|\le\left|p_{ij}(t)\right|+\left|q_{ij}(t)\right|\cdot\left|v_j(t)\right|\le \\ &\le p+q=a, \quad 0<a<+\infty.\end{aligned} \tag{6}$$

Theorem 1.1 Let $a\in R$ satisfies the inequality (6). Then, every nontrivial solution $y(t)$ of nonlinear differential system (3) satisfies the inequality

$$e^{-a(t-t_0)}\le\frac{\|y(t)\|}{\|y(t_0)\|}\le e^{a(t-t_0)}, \quad t\ge t_0. \tag{7}$$

Proof: Due to the fact that all constants *a* do not depend on the parameter *w*, there suffices to prove this theorem for all nontrivial solutions of (4).

In the first part of the proof we show that any nontrivial solution $y(t)$ of (4) satisfies the inequality

$$\left|\frac{d}{dt}\|y(t)\|^2\right|\le 2a\|y(t)\|^2 . \tag{8}$$

Make the modify of the left hand side of (8) gives

$$\left|\frac{d}{dt}\|y(t)\|^2\right|=\left|\frac{d}{dt}(y(t),y(t))\right|=\left|(y'(t),y(t))+(y(t),y'(t))\right|=2\left|(y'(t),y(t))\right|=$$

$$=2\left|\sum_{i=1}^{n}y_i{}'(t)\cdot y_i(t)\right|=2\left|\sum_{i=1}^{n}\left(\sum_{j=1}^{n}a_{w,ij}(t)\cdot y_j(t)\right)\cdot y_i(t)\right|\le 2a\left|(y(t),y(t))\right|=2a\|y(t)\|^2.$$

The first part of the proof is complete.

In the second part of the proof multiplying both sides of this inequality by $\left\|y[t]\right\|^{-2}$, one may obtain

$$-a \le \frac{d}{dt}\ln\left\|y(t)\right\| \le a \cdot \tag{9}$$

Integration of (9) gives

$$-a(t-t_0) \le \ln\frac{\left\|y(t)\right\|}{\left\|y(t_0)\right\|} \le a(t-t_0) \cdot$$

Consequently,

$$e^{-a(t-t_0)} \le \frac{\left\|y(t)\right\|}{\left\|y(t_0)\right\|} \le e^{a(t-t_0)} \cdot$$

Notice that the solution $w(t)$ satisfies the inequality (7), too.

The proof is complete.

Remark: Implicit in this theorem is the fact, that if $y(t)$ satisfies the inequality (7) then Lyapunov´s exponents satisfy the inequality

$$-a \le \omega \le \underline{\lambda} \le \overline{\lambda} \le \Omega \le a \cdot$$

Conclusion

Lyapunov's exponents are important in the study of a asymptotic behavior of solutions nonlinear differential equations. Nonlinear dynamical systems order greater than 2 have the capability of exhibiting a chaotic behavior that is highly complex. Lyapunov's exponents can be used to study a chaotic behavior of solutions of neurodynamical systems, too.

References

[1] Buša J., Pirč V.: Trajectory Bounds of Solutions of Certain Systems of Nonlinear Differential Equations, in Bull. Appl. Math., Baia mare, Romania, (2001), pp. 77-84

[2] Daňo I.: Compatible Neural Networks, in Journal of Electrical Engineering, 50 (1999), No. 7-8, pp. 203-205

[3] Daňo I.: The Central Exponent, in Zborník vedeckých prác Vysokej školy technickej v Košiciach, 1991, pp. 37-45

[4] Daňo I.: Pokazateli Liapunova, in Zborník 9. letnej školy z diferenciálnych rovníc, Poprad, 1986, pp. 21-24

[5] Daňo I.: Stability Theory by Liapunov's First Method and Neural Networks, (to appear)

[6] Daňo I., Haščák A.: Neurodynamické systémy, (to appear)

[7] Haščák A.: Disconjugacy and Multipoint Boundary Value Problems for Linear Differential Equations with Delay, in Czechoslovak Mathematical Journal, 39 (114) 1989, pp. 70-77

[8] Hopfield J. J., Tank D. W.: Neural computation of decisions in optimization problems, Biol. Cybern., 52 (1985), pp. 141-152

[9] Kolcun M., Szathmáry P.: Predikcia denných diagramov zaťaženia v ES s využitím umelých neurónových sietí, elfa s.r.o., Košice, 2001

[10] Liapunov A. M.: Obščaja zadača ob ustojčivosti dviženia, M. - L., Gostexizdat, 1950