On Rational Uninorms*

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Abstract: In this paper we characterize those uninorms which are rational functions (i.e., quotients of two polynomials). These are closely related to the well-known parametric families of t-norms and t-conorms studied and characterized by Hamacher [8].

Keywords: Associative operations, Uninorm, Representation, Rational uninorm, Hamacher family of t-norms and t-conorms.

1 Introduction

Uninorms were introduced by Yager and Rybalov [14] as a generalization of t-norms and t-conorms. For uninorms, the neutral element is not forced to be either 0 or 1, but can be any value e in the unit interval.

Definition 1. [14] A uninorm U is a commutative, associative and increasing binary operator with a neutral element $e \in [0,1]$, i.e. $U(e,x) = x$ holds for all $x \in [0,1]$.

Although this definition seems to be rather technical, we emphasize that there are practical reasons behind uninorms. The first comes from multicriteria decision making, where aggregation is one of the key issues. Suppose that some alternatives are evaluated from several points of view, and each evaluation is a number from the unit interval. Let us choose a level of satisfaction $e \in [0,1]$. Two semantical ways of expressing aggregation of the obtained numbers are as follows [14]:

(i) If **all** criteria are satisfied to at least e-extent then we are satisfied with any of them; else we want all of them satisfied.
(ii) If **any** of the evaluations is above e then we are satisfied with any of them; else we want them all satisfied.

Such situations can perfectly be modelled by uninorms and this leads to the particular classes introduced in [14], and also to the general forms studied in [6].

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The second reason that supports the practical use of uninorms comes from the field of expert systems. It is known (see e.g. [9]) that in MYCIN-like expert systems *c*ombining functions are used to calculate the global degrees of suggested diagnoses. A careful study reveals that such combining functions are *r*epresentable uninorms [3].

From a theoretical point of view, it is interesting to notice that uninorms U with a neutral element in $]0,1[$ are just those binary operators which make the structures $([0,1], \sup, U)$ and $([0,1], \inf, U)$ distributive semirings in the sense of Golan [7]. Further, in the theory of fuzzy measures and related integrals, uninorms play the role of pseudo-multiplication [12].

Therefore, the class of uninorms seems to play an interesting and important role both in theoretical investigations and in practical applications.

The main aim of the present paper is to characterize the class of uninorms that are quotients of two polinomials (in other words, *rational*). The root of this study goes back to the well-known results of Hamacher [8] about rational t-norms and t-conorms. We also study the link between the underlying t-norm and t-conorm of rational uninorms and members in the Hamacher family.

2 Preliminaries

2.1 Hamacher family of t-norms, t-conorms and negations

Let us define three parametrized families of t-norms, t-conorms and strong negations, respectively, as follows ($x, y \in [0,1]$).

$$\begin{aligned} T_\alpha(x,y) &= \frac{xy}{\alpha + (1-\alpha)(x+y-xy)}, \quad \alpha \geq 0, \\ S_\beta(x,y) &= \frac{x+y+(\beta-1)xy}{1+\beta xy}, \quad \beta \geq -1, \\ N_\gamma(x) &= \frac{1-x}{1+\gamma x}, \quad \gamma > -1. \end{aligned}$$

Hamacher [8] proved the following characterization theorem.

Theorem 1. (T, S, N) *is a De Morgan triplet such that*

$$\begin{aligned} T(x,y) = T(x,z) &\quad \Longrightarrow \quad y = z, \\ S(x,y) = S(x,z) &\quad \Longrightarrow \quad y = z, \\ \forall z \leq x \quad \exists y, y' & \text{ such that } T(x,y) = z, S(z,y') = x \end{aligned}$$

and T and S are rational functions if and only if there are numbers $\alpha \geq 0, \beta \geq -1$ and $\gamma > -1$ such that $\alpha = \frac{1+\beta}{1+\gamma}$ and $T = T_\alpha, S = S_\beta$ and $N = N_\gamma$. □

The above family of t-norms and t-conorms is called the *Hamacher family* of t-norms and t-conorms, respectively. Note that each member of these families is a strict t-norm resp. a strict t-conorm. Additive generators f_α of T_α are given as follows:

$$f_\alpha(x) = \begin{cases} \dfrac{1-x}{x} & \text{if } \alpha = 0 \\ \log\left(\frac{\alpha+(1-\alpha)x}{x}\right) & \text{if } \alpha > 0 \end{cases}.$$

Members of the Hamacher family of t-norms are decreasing functions of the parameter α.

Another characterization of the Hamacher family of t-norms with positive parameter has been obtained by Fodor and Keresztfalvi [5] as the only solutions of a functional equation

$$T\left(x, 1 - \frac{T(x,1-y)}{x}\right) = xy, \quad x,y \in]0,1],$$

where T is a t-norm such that the function φ defined by $\varphi(x) = \left[\frac{\partial T(x,y)}{\partial y}\right]_{y=0}$, $x \in [0,1]$, is a multiplicative generator of T (that is, $T(x,y) = \varphi^{-1}(\varphi(x)\varphi(y))$, $x, y \in [0,1]$).

2.2 Uninorms

It is well-known that t-norms do not allow low values to be compensated by high values, while t-conorms do not allow high values to be compensated by low values. Uninorms may allow values separated by their neutral element to be aggregated in a compensating way.

The structure of uninorms was studied by Fodor *et al.* [6]. For a uninorm U with neutral element $e \in]0,1]$, the binary operator T_U defined by

$$T_U(x,y) = \frac{U(e\,x, e\,y)}{e}$$

is a t-norm; for a uninorm U with neutral element $e \in [0,1[$, the binary operator S_U defined by

$$S_U(x,y) = \frac{U(e + (1-e)x, e + (1-e)y) - e}{1-e}$$

is a t-conorm. The structure of a uninorm with neutral element $e \in]0,1[$ on the squares $[0,e]^2$ and $[e,1]^2$ is therefore closely related to t-norms and t-conorms. For $e \in]0,1[$, we denote by ϕ_e and ψ_e the linear transformations defined by $\phi_e(x) = \frac{x}{e}$ and $\psi_e(x) = \frac{x-e}{1-e}$. To any uninorm U with neutral element $e \in]0,1[$, there corresponds a t-norm T and a t-conorm S such that:

(i) for any $(x,y) \in [0,e]^2$: $U(x,y) = \phi_e^{-1}(T(\phi_e(x), \phi_e(y)))$;

(ii) for any $(x,y) \in [e,1]^2$: $U(x,y) = \psi_e^{-1}(S(\psi_e(x), \psi_e(y)))$.

On the remaining part of the unit square, i.e. on

$$E = \big([0,e[\times]e,1]\big) \cup \big(]e,1] \times [0,e[\big),$$

it satisfies

$$\min(x,y) \leq U(x,y) \leq \max(x,y),$$

and could therefore partially show a compensating behaviour, i.e. take values strictly between minimum and maximum. Note that any uninorm U is either *conjunctive*, i.e. $U(0,1) = U(1,0) = 0$, or *disjunctive*, i.e. $U(0,1) = U(1,0) = 1$.

The classes of continuous t-norms and continuous t-conorms are well understood [11]. Recall that any continuous t-norm is either the minimum operator (the only idempotent t-norm), a continuous Archimedean t-norm (i.e. a t-norm with a continuous additive generator), or an ordinal sum of continuous Archimedean t-norms. The class of continuous Archimedean t-norms is subdivided into two disjoint classes: the class of strict t-norms and the class of nilpotent t-norms.

The purpose of the next section is to give an overview of representable uninorms.

3 Representable uninorms

In analogy to the representation of continuous Archimedean t-norms and t-conorms in terms of additive generators, Fodor *e*t al. [6] have investigated the existence of uninorms with a similar representation in terms of a single-variable function. This search leads back to Dombi's class of *aggregative operators* [4]. This work is also closely related to that of Klement *e*t al. on associative compensatory operators [10]. Consider $e \in]0,1[$ and a strictly increasing continuous $[0,1] \to \overline{\mathbb{R}}$ mapping h with $h(0) = -\infty$, $h(e) = 0$ and $h(1) = +\infty$. The binary operator U defined by

$$U(x,y) = h^{-1}(h(x) + h(y))$$

for any $(x,y) \in [0,1]^2 \setminus \{(0,1),(1,0)\}$, and either $U(0,1) = U(1,0) = 0$ or $U(0,1) = U(1,0) = 1$, is a uninorm with neutral element e. The class of uninorms that can be constructed in this way has been characterized [6]. Consider a uninorm U with neutral element $e \in]0,1[$, then there exists a strictly increasing continuous $[0,1] \to \overline{\mathbb{R}}$ mapping h with $h(0) = -\infty$, $h(e) = 0$ and $h(1) = +\infty$ such that

$$U(x,y) = h^{-1}(h(x) + h(y))$$

for any $(x,y) \in [0,1]^2 \setminus \{(0,1),(1,0)\}$ if and only if

(i) U is strictly increasing and continuous on $]0,1[^2$;
(ii) there exists an involutive negator N with fixpoint e such that

$$U(x,y) = N(U(N(x), N(y))))$$

for any $(x,y) \in [0,1]^2 \setminus \{(0,1),(1,0)\}$.

The uninorms characterized above are called *representable* uninorms. The mapping h is called an *additive generator* of U. The involutive negator corresponding to a representable uninorm U with additive generator h, as mentioned in condition (ii) above, is denoted N_U and is given by

$$N_U(x) = h^{-1}(-h(x)). \tag{1}$$

Clearly, any representable uninorm comes in a conjunctive and a disjunctive version, i.e., there always exist two representable uninorms that only differ in the points $(0,1)$ and $(1,0)$. Representable uninorms are almost continuous, i.e. continuous except in $(0,1)$ and $(1,0)$, and Archimedean, in the sense that $(\forall x \in]0,e[)(U(x,x) < x)$ and $(\forall x \in]e,1[)(U(x,x) > x)$.

A very important fact is that the underlying t-norm and t-conorm of a representable uninorm must be strict and cannot be nilpotent. Moreover, given a strict t-norm T with decreasing additive generator f and a strict t-conorm S with increasing additive generator g, we can always construct a representable uninorm U with desired neutral element $e \in]0,1[$ that has T and S as underlying t-norm and t-conorm. It suffices to consider as additive generator the mapping h defined by

$$h(x) = \begin{cases} -f\left(\dfrac{x}{e}\right) & \text{, if } x \leq e \\ g\left(\dfrac{x-e}{1-e}\right) & \text{, if } x \geq e \end{cases}. \tag{2}$$

As an example of the representable case, consider the additive generator h defined by $h(x) = \log \frac{x}{1-x}$, then the corresponding conjunctive representable uninorm $\mathcal{U}$ is given by

$$U(x,y) = \begin{cases} 0 & \text{, if } (x,y) \in \{(1,0),(0,1)\} \\ \dfrac{xy}{(1-x)(1-y)+xy} & \text{, elsewhere} \end{cases}$$

and has as neutral element $\frac{1}{2}$. Note that N_U is the standard negator: $N_U(x) = 1 - x$.

The class of representable uninorms contains famous operators, such as the functions for combining certainty factors in the expert systems MYCIN (see [13,3]) and PROSPECTOR [3]. The MYCIN expert system was one of the first systems capable of reasoning under uncertainty [2]. To that end, certainty factors were introduced as numbers in the interval $[-1,1]$. Essential in the processing of these certainty factors is the modified combining function C proposed by van Melle [2]. The $[-1,1]^2 \to [-1,1]$ mapping C is defined by

$$C(x,y) = \begin{cases} x + y(1-x) & \text{, if } \min(x,y) \geq 0 \\ \dfrac{x+y}{1-\min(|x|,|y|)} & \text{, if } \min(x,y) < 0 < \max(x,y) \\ x + y(1+x) & \text{, if } \max(x,y) \leq 0 \end{cases}.$$

The definition of C is not clear in the points $(-1,1)$ and $(1,-1)$, though it is understood that $C(-1,1) = C(1,-1) = -1$.

Rescaling the function C to a binary operator on $[0,1]$, we obtain a representable uninorm with neutral element $\frac{1}{2}$ and as underlying t-norm and t-conorm the product and the probabilistic sum. Implicitly, these results are contained in the book of Hájek *et al.* [9], in the context of ordered Abelian groups.

4 Rational uninorms

When U is a uninorm with neutral element $e \in]0,1[$ and with additive generator h, we have the following functional equation

$$h^{-1}(x+y) = U(h^{-1}(x), h^{-1}(y)) \tag{3}$$

for any $x, y \in \mathbb{R}$. This is obvious from the representation theorem.

Now we would like to study the following problem. Characterize those uninorms U with neutral element $e \in]0,1[$ such that

$$U(x,y) = \frac{P_n(x,y)}{P_m(x,y)}, \tag{4}$$

where P_n and P_m are polynomials of order n and m, respectively. We call a rational uninorm *proper* if its neutral element is strictly between 0 and 1.

Theorem 2. *Proper rational uninorms are given by the following parametric form for* $(x,y) \in [0,1]^2 \setminus \{(0,1),(1,0)\}$

$$U_e(x,y) = \frac{(1-e)xy}{(1-e)xy + e(1-x)(1-y)}, \tag{5}$$

and either $U(0,1) = U(1,0) = 0$ *or* $U(0,1) = U(1,0) = 1$, *where* $e \in]0,1[$ *is the neutral element of* U_e.

Proof. The general form of the inverses of generator functions of U having the form (4) is presented in [1, page 61]. According to that result, we have either

$$h^{-1}(x) = \frac{Ax+B}{Cx+D}, \tag{6}$$

or

$$h^{-1}(x)=\frac{A\exp{(cx)}+B}{C\exp{(cx)}+D}. \tag{7}$$

<u>Case 1: h^{-1} is of the form (6).</u>

Since we must have $h^{-1}(0)=e$, it follows that $B=De$. On the other hand, $\lim_{x\to+\infty}h^{-1}(x)=1$ implies $A=C$, $C\neq 0$. Then, however, we have

$$\lim_{x\to(-D/C)^+}h^{-1}(x)=\lim_{x\to(-D/C)^-}h^{-1}(x)=-\infty,$$

whence h^{-1} cannot be increasing on $\mathbb{R}$, neither continuous. Thus, there is no proper uninorm corresponding to (6).

<u>Case 2: h^{-1} is of the form (7).</u>

Then $h^{-1}(0)=e$ implies $\dfrac{A+B}{C+D}=e$.

When $c>0$ in (7), it follows from $\lim_{x\to+\infty}h^{-1}(x)=1$ that $A=C$, while $\lim_{x\to-\infty}h^{-1}(x)=0$ implies $B=0$, $D\neq 0$. Therefore, in this case we must have

$$h^{-1}(x)=\frac{e\exp{(cx)}}{e\exp cx+1-e}\qquad (x\in\mathbb{R}),$$

and

$$h(x)=\frac{1}{c}\ln\left(\frac{x-ex}{e-ex}\right)\qquad (x\in]0,1[).$$

When $c<0$ in (7), similar arguments lead to

$$h^{-1}(x)=\frac{e}{(1-e)\exp{(cx)}+e}\qquad (x\in\mathbb{R}),$$

and thus

$$h(x)=\frac{1}{c}\ln\left(\frac{e-ex}{x-ex}\right)\qquad (x\in]0,1[).$$

After some easy calculations we can come up with the formula in (5).

□

5 Relationship between the class of rational uninorms and the Hamacher family of t-norms and t-conorms

In this section we determine the underlying t-norm and t-conorm for the rational uninorm U_e in (5).

According to the general case explained in the previous subsection, both T_{U_e} and S_{U_e} have additive generators denoted by f and g, respectively. Moreover, we have by (2) that

$$f(x) = -h(ex) \qquad (x \in]0,1]), \tag{8}$$
$$g(x) = h(e + (1-e)x) \quad (x \in [0,1[). \tag{9}$$

Thus, we have

$$f(x) = -\frac{1}{c}\ln\left(\frac{1-ex}{x-ex}\right) \qquad (x \in]0,1]), \tag{10}$$
$$f^{-1}(x) = \frac{1}{e+(1-e)\exp(-cx)} \qquad (x \in [0,\infty[\,), \tag{11}$$

whence

$$T_{U_e}(x,y) = \frac{(1-e)xy}{1-e(x+y-xy)} \qquad (x,y \in [0,1]).$$

It is obvious that thus obtained T_{U_e} belongs to the Hamacher family of t-norms with parameter $\alpha = \dfrac{1}{1-e}$.

One can obtain in a similar way that

$$S_{U_e}(x,y) = \frac{ex+ey+(1-2e)xy}{e+(1-e)xy} \qquad (x,y \in [0,1]).$$

Therefore, S_{U_e} belogs to the Hamacher family of t-conorms with parameter $\beta = \dfrac{1-e}{e}$.

Finally, the parameter in the formula of the rational strong negation which stands in the De Morgan triplet is given by $\gamma = 2 - \frac{1}{e}$.

6 Conclusion

We have completely characterized rational t-norms. Relationships between the parameters of each rational uninorm and those of its underlying t-norm and t-conorm have also been determined.

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