t-Norm-based addition of fuzzy intervals *

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Abstract

The aim of this paper is to provide new results regarding the effective practical computation of t-norm-based additions of LR-fuzzy intervals.

Keywords: t-norm; extension principle; fuzzy interval

1 Introduction

The present paper is devoted to the derivation of exact calculation formulas for a t-norm-based addition of LR-fuzzy intervals (the addition rule of LR-fuzzy intervals is well-known in the case of "min"-norm). A very important feature of the approach by t-norms is that it provides means of controlling the growth of uncertainty in calculations, and prevents variables from simultaneous shift off their most significant values. In this respect, the various addition formulas yield practical tools for achieving this control, and are very meaningful. Our results are connected with those presented in [2] and we generalize and extend them. Namely, we shall determine the exact membership function of the t-norm-based sum of fuzzy intervals, in the case of Archimedean t-norm having strictly convex additive generator function and fuzzy intervals with concave shape functions. It should be noted that the class of t-norms mentioned above is wide enough (it contains among others: Yager's, Dombi's, Hamacher's, Schweizer's, Frank's, Weber's parametrized t-norms with adequate parameters) to be useful in practical computations.

2 Definitions

A fuzzy interval $\tilde{a}$ is a fuzzy set of real numbers $\mathbb{R}$ with a continuous, compactly supported, unimodal and normalized membership function. It is known [1], that any fuzzy interval $\tilde{a}$ can be described with the following membership function:

$$A(t) = \begin{cases} L\left(\frac{a-t}{\alpha}\right) & \text{if } t \in [a-\alpha, a] \\ 1 & \text{if } t \in [a, b] \\ R\left(\frac{t-b}{\beta}\right) & \text{if } t \in [b, b+\beta] \\ 0 & \text{otherwise} \end{cases}$$

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where $[a, b]$ is the peak of $\tilde{a}$; a and b are the lower and upper modal values; L and R are shape functions: $[0, 1] \rightarrow [0, 1]$, with $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$ which are non-increasing, continuous mappings. We shall call these fuzzy intervals of LR-type and use the notation $\tilde{a} = (a, b, \alpha, \beta)_{LR}$. The support of $\tilde{a}$ is exactly $[a - \alpha, b + \beta]$.

A function $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is said to be a triangular norm (t-norm for short) iff T is symmetric, associative, non-decreasing in each argument, and $T(x, 1) = x$ for all $x \in [0, 1]$. Recall that a t-norm T is Archimedean iff T is continuous and $T(x, x) < x$ for all $x \in]0, 1[$. Every Archimedean t-norm T is representable by a continuous and decreasing function $f : [0, 1] \rightarrow [0, \infty]$ with $f(1) = 0$ and

$$T(x, y) = f^{[-1]}(f(x) + f(y))$$

where $f^{[-1]}$ is the pseudo-inverse of f , defined by

$$f^{[-1]}(y) = \begin{cases} f^{-1}(y) & \text{if } y \in [0, f(0)] \\ 0 & \text{otherwise} \end{cases}$$

The function f is the additive generator of T . If T is a t-norm and $\tilde{a}_1, \tilde{a}_2$ are fuzzy sets of the real line (i.e. fuzzy quantities) then their T -sum $\tilde{A}_2 := \tilde{a}_1 + \tilde{a}_2$ is defined by

$$\tilde{A}_2(z) = \sup_{x_1 + x_2 = z} T(\tilde{a}_1(x_1), \tilde{a}_2(x_2)), \quad z \in \mathbb{R}$$

which expression can be written in the form

$$\tilde{A}_2(z) = f^{[-1]}(f(\tilde{a}_1(x_1)) + f(\tilde{a}_2(x_2)))$$

supposing that f is the additive generator of T .

By the associativity of T , the membership function of the T -sum $\tilde{A}_n := \tilde{a}_1 + \dots + \tilde{a}_n$ of fuzzy quantities $\tilde{a}_1, \dots, \tilde{a}_n$ can be written as

$$\tilde{A}_n(z) = \sup_{x_1 + \dots + x_n = z} f^{[-1]} \left(\sum_{i=1}^n f(\tilde{a}_i(x_i)) \right).$$

Since f is continuous and decreasing, $f^{[-1]}$ is also continuous and non-increasing, we have

$$\tilde{A}_n(z) = f^{[-1]} \left(\inf_{x_1 + \dots + x_n = z} \sum_{i=1}^n f(\tilde{a}_i(x_i)) \right).$$

3 t-norm-based addition of fuzzy intervals

In the following theorem we shall determine a class of t-norms in which the addition of fuzzy intervals is very simple.

Theorem 3.1 *Let T be an Archimedean t-norm with additive generator f and let $\tilde{a}_i = (a_i, b_i, \alpha, \beta)_{LR}$, $i = 1, \dots, n$, be fuzzy numbers of LR-type. If L and R are twice differentiable, concave functions, and f is twice differentiable, strictly convex function then the membership function of the T -sum $\tilde{A}_n =$*

$\tilde{a}_1 + \dots + \tilde{a}_n$ is

$$\tilde{A}_n(z) = \begin{cases} 1 & \text{if } A_n \leq z \leq B_n \\ f^{[-1]} \left(n \times f \left(L \left(\frac{A_n - z}{n\alpha} \right) \right) \right) & \text{if } A_n - n\alpha \leq z \leq A_n \\ f^{[-1]} \left(n \times f \left(R \left(\frac{z - B_n}{n\beta} \right) \right) \right) & \text{if } B_n \leq z \leq B_n + n\beta \\ 0 & \text{otherwise} \end{cases}$$

where $A_n = a_1 + \dots + a_n$ and $B_n = b_1 + \dots + b_n$.

Proof 1 As it was mentioned above, the investigated membership function is

$$\tilde{A}_n(z) = f^{[-1]} \left(\inf_{x_1 + \dots + x_n = z} f(\tilde{a}_1(x_1)) + \dots + f(\tilde{a}_n(x_n)) \right). \quad (1)$$

It is easy to see that the support of $\tilde{A}_n$ is included in the interval $[A_n - n\alpha, B_n + n\beta]$. From the decomposition rule of fuzzy numbers into two separate parts it follows that the peak of $\tilde{A}_n$ is $[A_n, B_n]$. Moreover, if we consider the right hand side of $\tilde{A}_n$ (i.e. $B_n \leq z \leq B_n + n\beta$) then only the right hand sides of terms $\tilde{a}_n$ come into account in (1) (i.e. $b_i \leq x_i \leq b_i + \beta$, $i = 1, \dots, n$). The same thing holds for the left hand side of $\tilde{A}_n$ (i.e. if $A_n - n\alpha \leq z \leq A_n$ then $a_i - \alpha \leq x_i \leq a_i$, $i = 1, \dots, n$).

Let us now consider the right hand side of $\tilde{A}_n$, so let $B_n \leq z \leq B_n + n\beta$. (A similar method can be used if $A_n - n\alpha \leq z \leq A_n$.) The constraints

$$x_1 + \dots + x_n = z, \quad b_i \leq x_i \leq b_i + \beta, \quad i = 1, \dots, n$$

determine a compact and convex domain $\mathcal{K} \subset \mathbb{R}^n$ which can be considered as the section of the brick

$$\mathcal{B} := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid b_i \leq x_i \leq b_i + \beta, \quad i = 1, \dots, n\}$$

by the hyperplane

$$\mathcal{P} := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 + \dots + x_n = z\}$$

In order to determine $\tilde{A}_n(z)$ we need to find the conditional minimum value of the function $\phi : \mathcal{B} \rightarrow \mathbb{R}$,

$$\phi(x_1, \dots, x_n) := f(\tilde{a}_1(x_1)) + \dots + f(\tilde{a}_n(x_n))$$

subject to condition $(x_1, \dots, x_n) \in \mathcal{K}$. $\mathcal{K}$ is compact and ϕ is continuous, this is why we could change the infimum with minimum.

Following the Lagrange's multipliers method we are searching the stationary points of the auxiliary function $\Phi : \mathcal{B} \rightarrow \mathbb{R}$

$$\Phi(x_1, \dots, x_n) := \phi(x_1, \dots, x_n) + \lambda(z - (x_1 + x_2 + \dots + x_n))$$

i.e. the points $(x_1, \dots, x_n, \lambda) \in \mathcal{B} \times \mathbb{R}$ where its derivative vanishes.

It is clear, that Φ is twice differentiable, and its partial derivative with respect to x_i is

$$\partial_i \Phi(x_1, \dots, x_n, \lambda) = f'(R(\sigma_i(x_i)))R'(\sigma_i(x_i))\frac{1}{\beta} - \lambda,$$

where $\sigma_i(x_i) := (x_i - b_i)/\beta$ and with respect to λ is

$$\partial_\lambda \Phi(x_1, \dots, x_n, \lambda) = z - (x_1 + \dots + x_n)$$

We get the system of equations:

$$\begin{aligned} f'(R(\sigma_i(x_i)))R'(\sigma_i(x_i))\frac{1}{\beta} - \lambda &= 0, \quad i = 1, \dots, n, \\ z - (x_1 + \dots + x_n) &= 0. \end{aligned}$$

If we take

$$\hat{x}_i = b_i + \frac{z - B_n}{n}, \quad i = 1, \dots, n,$$

then

$$\sigma_1(x_1) = \dots = \sigma_n(x_n)$$

and we can define

$$\lambda = f'(R(\sigma_i(x_i)))R'(\sigma_i(x_i))\frac{1}{\beta}.$$

It is easy to verify that $(\hat{x}_1, \dots, \hat{x}_n, \lambda)$ is a solution of the system of equations above.

This means that ϕ attains its absolute conditional minimum at the point $\hat{x}$. Thus we have obtained a stationary point of the auxiliary function Φ . It is left to show, that $\Phi|_{\mathcal{K}}$ attains its absolute minimum at $\hat{x} := (\hat{x}_1, \dots, \hat{x}_n) \in \mathcal{K}$. For, then

$$\tilde{A}_n(z) = f^{[-1]}(f(\tilde{a}_1(\hat{x}_1)) + \dots + f(\tilde{a}_n(\hat{x}_n))) =$$

$$f^{[-1]}(f(R(\sigma_1(\hat{x}_1))) + \dots + f(R(\sigma_n(\hat{x}_n)))) =$$

$$f^{[-1]} \left(n \cdot f \left(R \left(\frac{z - B_n}{n\beta} \right) \right) \right).$$

It is easy to see that the partial derivatives of the second order of ϕ at any point $x = (x_1, \dots, x_n) \in \mathcal{B}$ have the following properties:

$$\partial_{ij}\phi(x) = 0, \quad \text{if } i \neq j$$

and for $i = 1, \dots, n$,

$$\partial_{ii}\phi(x) = \frac{1}{\beta^2} [f''(R(\sigma_i(x_i)))(R'(\sigma_i(x_i)))^2 + f'(R(\sigma_i(x_i)))R''(\sigma_i(x_i))].$$

Therefore, we have $\partial_{ii}(x) > 0$ for each $i = 1, \dots, n$. Indeed,

- $R'(\sigma_i(x_i)) \neq 0$ since R is non-increasing and concave, hence strictly decreasing in a neighbourhood of $\sigma_i(x_i)$;
- $f' < 0$, $f'' > 0$ and $R'' < 0$ hold by monotonicity and strict convexity of f and concavity of R .

The matrix of the derivative of the second order of ϕ at any point inside $\mathcal{B}$ has only (nonzero) elements in its diagonal, which are positive. Therefore, it is positive definite in $\mathcal{B}$. Now we show that $\phi(\hat{x})$ is the minimum of ϕ in $\mathcal{K}$.

Consider an arbitrary point $x = (x_1, \dots, x_n) \in \mathcal{K}$. From convexity of $\mathcal{K}$ it follows that the segment $[\hat{x}, x]$ lies within $\mathcal{K}$. By virtue of Lagrange's mean value theorem, there exists $\xi = (\xi_1, \dots, \xi_n) \in [\hat{x}, x]$ such that

$$\phi(x) = \phi(\hat{x}) + \sum_{i=1}^n \partial_i \phi(\hat{x})(x_i - \hat{x}_i) + \sum_{i,j=1}^n \partial_{ij} \phi(\xi)(x_i - \hat{x}_i)(x_j - \hat{x}_j)$$

and using the properties $\partial_1 \phi(\hat{x}) = \dots = \partial_n \phi(\hat{x}) = 0$, $\partial_{ij} \phi(\hat{x}) = 0$, if $i \neq j$ and $\partial_{ii} \phi(\hat{x}) > 0$, for each i , we obtain that $\phi(x) > \phi(\hat{x})$. This means that ϕ attains its absolute conditional minimum at the point $\hat{x}$.

Remark 3.1 It should be noted, that from the concavity of shape functions it follows that the fuzzy intervals in question can not have infinite support.

4 Applications

We shall illustrate Theorem 1 for Yager's, Dombi's and Hamacher's parametrized t-norm.

For simplicity we shall restrict our consideration to the case of symmetric fuzzy numbers $\tilde{a}_i = (a_i, a_i, \alpha, \alpha)_{LL}$, $i = 1, \dots, n$. Denoting

$$\sigma_n := \frac{|A_n - z|}{n\alpha}$$

we get the following formulas for the membership function of t-norm-based sum $\tilde{A}_n = \tilde{a}_1 + \dots + \tilde{a}_n$:

(i) Yager's t-norm with $p > 1$:

$$T(x, y) = 1 - \min \left\{ 1, \sqrt[p]{(1-x)^p + (1-y)^p} \right\}.$$

This has additive generator

$$f(x) = (1-x)^p$$

and then

$$\tilde{A}_n(z) = \begin{cases} 1 - n^{1/p}(1 - L(\sigma_n)) & \text{if } \sigma_n < L^{-1}(1 - n^{-1/p}) \\ 0 & \text{otherwise.} \end{cases}$$

(ii) Hamacher's t-norm with $p \leq 2$ (see [7] for $n = 2$):

$$T(x, y) = \frac{xy}{p + (1-p)(x + y - xy)}$$

having additive generator

$$f(x) = \ln \frac{p + (1-p)x}{x}$$

Then

$$\tilde{A}_n(z) = \begin{cases} \frac{p}{[(p + (1-p)L(\sigma_n))/L(\sigma_n)]^n - 1 + p} & \text{if } \sigma_n < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(iii) Dombi's t-norm with $p > 1$:

$$T(x, y) = \frac{1}{1 + \sqrt[p]{(1/x - 1)^p + (1/y - 1)^p}}$$

with additive generator

$$f(x) = \left(\frac{1}{x} - 1\right)^p.$$

Then

$$\tilde{A}_n(z) = \begin{cases} [1 + n^{1/p}(1/L(\sigma_n) - 1)]^{-1} & \text{if } \sigma_n < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(iv) Algebraic t-norm (i.e. the Hamacher's t-norm with $p = 1$):

$$T(x, y) = xy \text{ having additive generator } f(x) = -\ln x$$

Then

$$\tilde{A}_n(z) = L^n(\sigma_n), \quad z \in \mathbb{R}.$$

References

- [1] D.Dubois and H.Prade, *Fuzzy Sets and Systems: Theory and Applications*, Academic Press, New York, 1980.
- [2] D.Dubois and H.Prade, Additions of interactive fuzzy numbers, *IEEE Transactions on Automatic Control*, Vol. AC-26, No.4 1981, 926-936.
- [3] D.Dubois and H.Prade, Inverse Operations for Fuzzy Numbers, in: *Proc. of IFAC Symp. on Fuzzy Information, Knowledge, Representation and Decision Analysis*, Marseille, 1983 399-404.

5 Follow ups

The results of this paper have been improved and generalized later in the following works:

in journals

A24-c41 Dug Hun Hong, A NOTE ABOUT OPERATIONS LIKE T-W (THE WEAKEST t-NORM) BASED ADDITION ON FUZZY INTERVALS, *KYBERNETIKA*, 45 (3): 541-547 2009

<http://www.kybernetika.cz/content/2009/3/541>

A24-c40 Claudio De Capua, Emilia Romeo, A t-Norm-Based Fuzzy Approach to the Estimation of Measurement Uncertainty, *IEEE TRANSACTIONS ON INSTRUMENTATION AND MEASUREMENT*, 58(2009), pp.350-355. 2009

<http://dx.doi.org/10.1109/TIM.2008.2003339>

A24-c39 Janusz T. Starczewski, Extended triangular norms, INFORMATION SCIENCES, 179(2009), pp. 742-757. 2009
<http://dx.doi.org/10.1016/j.ins.2008.11.009>

This result was inspired by works [A24,12] on the addition of fuzzy intervals. (page 752)

A24-c38 Dug Hun Hong, A convexity problem and a new proof for linearity preserving additions of fuzzy intervals, FUZZY SETS AND SYSTEMS, 159(2008) pp. 3388-3392. 2008
<http://dx.doi.org/10.1016/j.fss.2008.05.020>

A24-c37 Hong DH, Hwang C, Kim KT, T-sum of sigmoid-shaped fuzzy intervals INFORMATION SCIENCES 177 (18): 3831-3839 SEP 15 2007
<http://dx.doi.org/10.1016/j.ins.2007.03.023>

A24-c36 Hong DH, T-sum of bell-shaped fuzzy intervals, FUZZY SETS AND SYSTEMS 158 (7): 739-746 APR 1 2007
<http://dx.doi.org/10.1016/j.fss.2006.10.021>

A24-c35 József Dombi and Norbert Györfi, Addition of sigmoid-shaped fuzzy intervals using the Dombi operator and infinite sum theorems, FUZZY SETS AND SYSTEMS, 157(2006) 952-963. 2006
<http://dx.doi.org/10.1016/j.fss.2005.09.011>

Fullér has studied the sup-T sum with triangular fuzzy intervals [A32,A30] and in a more general context [A24]. These results were developed further and extended by Hong [11-13] and Mesiar [15]. (page 953)

A24-c34 Hong DH On types of fuzzy numbers under addition KYBERNETIKA, 40 (4): 469-476. 2004
http://kybernetika.utia.cas.cz/pdf_article/40_4_654_full.pdf

A24-c33 Dug Hun Hong, T-sum of L-R fuzzy numbers with unbounded supports, Commun. Korean Math. Soc., 18 (2003), no. 2, 385–392. 2003

The next result on the sum of L-R fuzzy numbers based on an Archimedean continuous t-norm T is due to Fullér and Keresztfalvi [A24]. (page 387)

A24-c32 Dug Hun Hong, On shape-preserving additions of fuzzy intervals, JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS, 267 (1): 369-376 MAR 1 2002

A24-c31 Dug Hun Hong, Some results on the addition of fuzzy intervals, FUZZY SETS AND SYSTEMS, 122(2001) 349-352. 2001
[http://dx.doi.org/10.1016/S0165-0114\(00\)00005-1](http://dx.doi.org/10.1016/S0165-0114(00)00005-1)

A24-c30 Seok Yoon Hwang and Hyo Sam Lee, Nilpotent t-norm-based sum of fuzzy intervals, FUZZY SETS AND SYSTEMS, 123(2001) 73-80. 2001
[http://dx.doi.org/10.1016/S0165-0114\(01\)00005-7](http://dx.doi.org/10.1016/S0165-0114(01)00005-7)

- A24-c29** Dug Hun Hong, Shape preserving multiplications of fuzzy numbers, FUZZY SETS AND SYSTEMS, 123(2001) 81-84. 2001
[http://dx.doi.org/10.1016/S0165-0114\(00\)00107-X](http://dx.doi.org/10.1016/S0165-0114(00)00107-X)
- A24-c28** Klement EP, Mesiar R, Pap E Generated triangular norms KYBERNETIKA, 36 (3): 363-377. 2000
<http://dml.cz/dmlcz/135356>
- A24-c27** A.Kolesárová, Triangular norm-based addition preserving linearity of T-sums of linear fuzzy intervals, MATHWARE & SOFT COMPUTING, 5(1998) 91-98. 1998
- A24-c26** Hwang SY, Hwang JJ, An JH The triangular norm-based addition of fuzzy intervals APPLIED MATHEMATICS LETTERS, 11 (4): 9-13 JUL 1998
[http://dx.doi.org/10.1016/S0893-9659\(98\)00048-2](http://dx.doi.org/10.1016/S0893-9659(98)00048-2)
- A24-c25** B. De Baets and A. Marková-Stupňanová, Analytical expressions for addition of fuzzy intervals, FUZZY SETS AND SYSTEMS, 91(1997) 203-213. 1997
[http://dx.doi.org/10.1016/S0165-0114\(97\)00141-3](http://dx.doi.org/10.1016/S0165-0114(97)00141-3)

The first results for an arbitray continuous Archimedean t-norm are due to Fullér and Keresztfalvi [A24].

Proposition 7. (Fullér and Keresztfalvi [A24]). Consider a continuous Archimedean t-norm $\mathcal{T}$ with additive generator f and n LR-fuzzy intervals $A_i = (l_i, r_i, \alpha, \beta)_{LR}$, $i = 1, \dots, n$. If f is twice differentiable and strictly convex, and L and R are twice differentiable and concave, then the $\mathcal{T}$ -sum $\bigoplus_{\mathcal{T}, i=1}^n A_i$ is given by

$$\bigoplus_{\mathcal{T}, i=1}^n A_i(x) = \begin{cases} f^{(-1)} \left(n f \left(L \left(\frac{l-x}{n\alpha} \right) \right) \right) & \text{if } l - n\alpha \leq x \leq l, \\ 1 & \text{if } l \leq x \leq r, \\ f^{(-1)} \left(n f \left(R \left(\frac{x-r}{n\beta} \right) \right) \right) & \text{if } r \leq x \leq r + n\beta, \\ 0 & \text{elsewhere,} \end{cases}$$

where $l = \sum_{i=1}^n l_i$ and $r = \sum_{i=1}^n r_i$. (page 207)

- A24-c24** R.Mesiar, Triangular-norm-based addition of fuzzy intervals, FUZZY SETS AND SYSTEMS, 91(1997) 231-237. 1997
[http://dx.doi.org/10.1016/S0893-9659\(98\)00048-2](http://dx.doi.org/10.1016/S0893-9659(98)00048-2)
- A24-c23** D.H.Hong and C. Hwang, A T-sum bound of LR-fuzzy numbers, FUZZY SETS AND SYSTEMS, 91(1997) 239-252. 1997

The exact output of a $\mathcal{T}$ -sum of some LR-fuzzy numbers in an analytical form was found only for some special cases. For Archimedean t-norm $\mathcal{T}$ with additive generator f , the most general published result is due to Hong and Hwang [9], generalizing the earlier results of Fullér and Keresztfalvi [A24]. (page 240)

A24-c22 A. Marková-Stupňanová, Idempotents of T-addition of fuzzy numbers, *Tatra Mountains Mathematical Publications* 12(1997), 65-72. [MR: 98k:04005] 1997

<http://tatra.mat.savba.sk/Full/12/12markov.ps>

A24-c21 D.H.Hong and S.Y.Hwang, The convergence of T-product of fuzzy numbers, *FUZZY SETS AND SYSTEMS*, 85(1997) 373-378. 1997

[http://dx.doi.org/10.1016/0165-0114\(95\)00333-9](http://dx.doi.org/10.1016/0165-0114(95)00333-9)

Recently, t-norm-based addition of fuzzy numbers and its convergence have been studied [A32, A30, A28, A24, 6, 7, 9]. (page 373)

A24-c20 A.Markova, T-sum of L-R fuzzy numbers, *FUZZY SETS AND SYSTEMS*, 85(1997) 379-384. 1997

[http://dx.doi.org/10.1016/0165-0114\(95\)00370-3](http://dx.doi.org/10.1016/0165-0114(95)00370-3)

The next result on the sum of L-R fuzzy numbers based on an Archimedean continuous t-norm T is due to Fullér and Keresztfalvi [A24]. (page 380)

A24-c19 R.Mesiar, Shape preserving additions of fuzzy intervals, *FUZZY SETS AND SYSTEMS*, 86 73-78. 1997

[http://dx.doi.org/10.1016/0165-0114\(95\)00401-7](http://dx.doi.org/10.1016/0165-0114(95)00401-7) [10303873]

Some results on this topic and applications can be found e.g. in [2, A24-7, 11-13].

A24-c18 S.Y.Hwang and D.H.Hong, The convergence of T-sum of fuzzy numbers on Banach spaces, *APPLIED MATHEMATICS LETTERS*, 10 No. 4, 129-134. 1997

[http://dx.doi.org/10.1016/S0893-9659\(97\)00072-4](http://dx.doi.org/10.1016/S0893-9659(97)00072-4)

Recently, Houg and Hwang [1] determined the exact membership function of the t-norm-based sum of fuzzy numbers, in the case of Archimedean t-norm having convex additive generator function and fuzzy numbers with concave shape functions, which is the generalization of Fullér and Keresztfalvi's result [A24]. (page 129)

A24-c17 R.Mesiar, A note to the T-sum of L-R fuzzy numbers, *FUZZY SETS AND SYSTEMS*, 79(1996) 259-261. 1996

[http://dx.doi.org/10.1016/0165-0114\(95\)00178-6](http://dx.doi.org/10.1016/0165-0114(95)00178-6)

A24-c16 A. Kolesárová, Triangular norm-based addition of linear fuzzy numbers, *Tatra Mountains Mathematical Publications*, 6(1995) 75-81. [96g:04008] 1995

http://tatra.mat.savba.sk/paper.php?id_paper=193

A24-c15 D.H.Hong, A note on t-norm-based addition of fuzzy intervals, *FUZZY SETS AND SYSTEMS*, 75(1995) 73-76. 1995

[http://dx.doi.org/10.1016/0165-0114\(94\)00329-6](http://dx.doi.org/10.1016/0165-0114(94)00329-6)

We generalize a result of Fullér and Keresztfalvi (1992) regarding the computation of t-norm-based additions of LR-fuzzy intervals, the proof of which is new and simple. (page 73)

A24-c14 A.Markova, Addition of L-R fuzzy numbers, *Bulletin for Studies and Exchanges on Fuzziness and its Applications*, 63(1995) 25-29. 1995

A24-c13 D.H.Hong and S.Y.Hwang, On the convergence of T-sum of L-R fuzzy numbers, *FUZZY SETS AND SYSTEMS*, 63(1994) 175-180. 1994
[http://dx.doi.org/10.1016/0165-0114\(94\)90347-6](http://dx.doi.org/10.1016/0165-0114(94)90347-6)

A24-c12 D.H.Hong and S.Y.Hwang, On the compositional rule of inference under triangular norms, *FUZZY SETS AND SYSTEMS*, 66(1994) 25-38. 1994
[http://dx.doi.org/10.1016/0165-0114\(94\)90299-2](http://dx.doi.org/10.1016/0165-0114(94)90299-2)

A24-c11 M.F.Kawaguchi and T.Da-te, Some algebraic properties of weakly non-interactive fuzzy numbers, *FUZZY SETS AND SYSTEMS* 68(1994) 281-291. 1994
[http://dx.doi.org/10.1016/0165-0114\(94\)90184-8](http://dx.doi.org/10.1016/0165-0114(94)90184-8)

Nevertheless, we expect that the concrete characteristics of each t-norm would make it possible to cope with such a problem to a certain extent. For instance, Fullér et al. illustrated the sums of triangular fuzzy numbers based on various t-norms: algebraic product, Hamacher product, Yager's parametrized t-norm, etc. [A32, A30, A24]. Their results help us to evaluate a fuzzy linear expression of which either coefficients or variables are fuzzy numbers. (page 290)

A24-c10 M.F.Kawaguchi and T.Da-te, Properties of fuzzy arithmetic based on triangular norms, *Journal of Japan Society for Fuzzy Theory and Systems*, 5(1993) 1113-1121 (in Japanese). *Japanese Journal of Fuzzy Theory and Systems*, 5(1993) 677-687 (English Translation version). 1993

in proceedings and in edited volumes

A24-c9 Claudio De Capua and Emilia Romeo, A t-Norm Based Fuzzy Approach to the Estimation of Measurement Uncertainty, Instrumentation and Measurement Technology Conference (IMTC 2004) Como, Italy, 18-20 May 2004, [doi 10.1109/IMTC.2004.1351034], pp. 229-233. 2004
<http://dx.doi.org/10.1109/IMTC.2004.1351034>

A24-c8 D.Dubois, E.Kerre, R.Mesiar and H.Prade, Fuzzy interval analysis, in: Didier Dubois and Henri Prade eds., *Fundamentals of Fuzzy Sets*, The Handbooks of Fuzzy Sets Series, Volume 7, Kluwer Academic Publishers, [ISBN 0-7923-7732-X], 2000, 483-581. 2000

A closed form solution of the fuzzy addition in the case of fuzzy intervals was provided by Dubois and Prade (1981) for the three basic triangular norms other than the minimum. Several other results were obtained since then by Fullér and Keresztfalvi (1992), Kawaguchi and Da-te (1993, 1994), ... (page 526)

A24-c7 A. Marková-Stupňanová, Idempotent fuzzy intervals, in: *Proceedings of IPMU'98 Conference*, (July 6-10, 1998, Paris, La Sorbonne), [ISBN 2-842-54-013-1], 1998 255-258. 1998

We recall a formula for the T -sum based on the continuous Archimedean t-norms for special type of L-R fuzzy numbers introduced by Fullér and Keresztfalvi [A24] and generalized by Mesiar ... (page 257)

A24-c6 D.H.Hong and C. Hwang, Upper bound of T-sum of LR-fuzzy numbers, in: *Proceedings of IPMU'96 Conference*, (July 1-5, 1996, Granada, Spain), 1996 343-346. 1996

A24-c5 B. De Baets and A. Markova, Addition of LR-fuzzy intervals based on a continuous t-norm, in: *Proceedings of IPMU'96 Conference*, (July 1-5, 1996, Granada, Spain), 1996 353-358. 1996

A24-c4 M.F.Kawaguchi and T. Da-Te, A calculation method for solving fuzzy arithmetic equations with triangular norms, in: *Proceedings of Second IEEE international Conference on Fuzzy Systems*, 1993 470-476. 1993

<http://ieeexplore.ieee.org/iel2/1022/7759/00327513.pdf?>

in books

A24-c3 E. P. Klement, Radko Mesiar, Endre Pap, *Triangular Norms*, Springer, [ISBN 0792364163], 2000.

A24-c2 Jorma K. Mattila, *Text Book of Fuzzy Logic*, Art House, Helsinki, [ISBN 951-884-152-7], 1998.

A24-c1 R. Lowen, *Fuzzy Set Theory*, Kluwer Academic Publishers, [ISBN: 0-7923-4057-4], 1996.