An improved index of interactivity for fuzzy numbers∗

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Abstract

In this paper we will introduce a new index of interactivity between marginal possibility distributions $A$ and $B$ of a joint possibility distribution $C$. The starting point of our approach is to equip each $\gamma$-level set of $C$ with a uniform probability distribution, then the probabilistic correlation coefficient between its marginal probability distributions is interpreted as an index of interactivity between the $\gamma$-level sets of $A$ and $B$. Then we define the index of interactivity between $A$ and $B$ as the weighted average of these indexes over the set of all membership grades. This new index of interactivity is meaningful for the whole family of joint possibility distributions.

1 Introduction

In probability theory the notion of expected value of functions of random variables plays a fundamental role in defining the basic characteristic measures of probability distributions. For instance, the measure of covariance, variance and correlation of random variables can all be computed as probabilistic means of their appropriately chosen real-valued functions. For expected value, variance, covariance and correlation of fuzzy random variables the reader can consult, e.g. Kwakernaak [13, 14], Puri and Ralescu [18], Körner [15], Watanabe and Imaizumi [22], Feng et al [8], Näther [17] and Shapiro [19].

In possibility theory we can use the principle of average value of appropriately chosen real-valued functions to define mean value, variance, covariance and correlation of possibility distributions. A function $f$: $[0, 1] \rightarrow \mathbb{R}$ is said to be a weighting function if $f$ is non-negative, monotone increasing and satisfies the

following normalization condition \( \int_{0}^{1} f(\gamma) d\gamma = 1 \). Different weighting functions can give different (case-dependent) importances to level-sets of possibility distributions. We can define the mean value (variance) of a possibility distribution as the \( f \)-weighted average of the probabilistic mean values (variances) of the respective uniform distributions defined on the \( \gamma \)-level sets of that possibility distribution. A measure of possibilistic covariance between marginal possibility distributions of a joint possibility distribution can be defined as the \( f \)-weighted average of possibilistic covariances between marginal probability distributions whose joint probability distribution is defined to be uniform on the \( \gamma \)-level sets of their joint possibility distribution [10]. This is an absolute measure of interactivity. A measure of possibilistic correlation between marginal possibility distributions of a joint possibility distribution can be defined as their possibilistic covariance divided by the square root of the product of their possibilistic variances [2]. This is a relative measure of interactivity. We should note here that the choice of uniform probability distribution on the level sets of possibility distributions is not without reason. We suppose that each point of a given level set is equally possible and then we apply Laplace’s principle of Insufficient Reason: if elementary events are equally possible, they should be equally probable (for more details and generalization of principle of Insufficient Reason see [7], page 59). The idea of equipping the alpha-cuts with a uniform probability is not new and refers to early ideas of simulation of fuzzy sets by Yager [23], and possibility/probability transforms by Dubois et al [5] as well as the pignistic transform of Smets [20].

The main drawback of the measure of possibilistic correlation introduced in [2] that it does not necessarily take its values from \([-1, 1]\) if some level-sets of the joint possibility distribution are not convex. A new normalization technique is needed.

In this paper we will introduce a new index of interactivity between marginal distributions of a joint possibility distribution, which is defined for the whole family of joint possibility distributions. Namely, we will equip each level set of a joint possibility distribution with a uniform probability distribution, then compute the probabilistic correlation coefficient between its marginal probability distributions, and then the new index of interactivity is computed as the weighted average of these coefficients over the set of all membership grades. These weights (or importances) can be given by weighting functions.

A fuzzy number \( A \) is a fuzzy set in \( \mathbb{R} \) with a normal, fuzzy convex and continuous membership function of bounded support. The family of fuzzy numbers is denoted by \( \mathcal{F} \). Fuzzy numbers can be considered as possibility distributions. A \( \gamma \)-level set of a fuzzy set \( A \) in \( \mathbb{R}^m \) is defined by \( [A]^{\gamma} = \{ x \in \mathbb{R}^m : A(x) \geq \gamma \} \) if \( \gamma > 0 \) and \( [A]^{0} = \text{cl}\{ x \in \mathbb{R}^m : A(x) > \gamma \} \). A joint possibility distribution of fuzzy numbers is defined as a normal fuzzy set \( C \) in \( \mathbb{R}^2 \). Furthermore, \( A \) and \( B \)
are called the marginal possibility distributions of $C$ if it satisfies the relationships

$$\max\{y \in \mathbb{R} \mid C(x, y)\} = A(x) \text{ and } \max\{x \in \mathbb{R} \mid C(x, y)\} = B(y),$$

for all $x, y \in \mathbb{R}$. In the following we will suppose that $C$ is given in such a way that a uniform distribution can be defined on $[C]_\gamma$ for all $\gamma \in [0, 1]$. Marginal possibility distributions are always uniquely defined from their joint possibility distribution by the principle of falling shadows.

Let $C$ be a joint possibility distribution with marginal possibility distributions $A, B \in \mathcal{F}$, and let $[A]_\gamma = [a_1(\gamma), a_2(\gamma)]$ and $[B]_\gamma = [b_1(\gamma), b_2(\gamma)]$, $\gamma \in [0, 1]$. Then $A$ and $B$ are said to be non-interactive if their joint possibility distribution is $A \times B$,

$$C(x, y) = \min\{A(x), B(y)\},$$

for all $x, y \in \mathbb{R}$, which can be written in the form, $[C]_\gamma = [A]_\gamma \times [B]_\gamma$, that is, $[C]_\gamma$ is a rectangular subset of $\mathbb{R}^2$, for any $\gamma \in [0, 1)$. If $A$ and $B$ are are non-interactive then for any $x \in [A]_\gamma$ and any $y \in [B]_\gamma$ we have that the ordered pair $(x, y)$ will be in $[C]_\gamma$ for any $\gamma \in [0, 1]$. In other words, if one takes a point, $x$, from the $\gamma$-level set of $A$ and then takes an arbitrarily chosen point, $y$, from the $\gamma$-level set of $B$ then the pair $(x, y)$ will belong to the $\gamma$-level set of $C$.

Another extreme situation is when $[C]_\gamma$ is a line segment in $\mathbb{R}^2$. For example, let $[0, 1] \times [0, 1]$ be the universe of discourse for $C$ and let, the diagonal beam,

$$C(x, y) = x_{\chi_{x=y}}(x, y),$$

for any $x, y \in [0, 1]$, be the joint possibility distribution of marginal possibility distributions $A(x) = x$ and $B(y) = y$. Then $[C]_\gamma$ is a line segment $[(\gamma, \gamma), (1, 1)]$ in $\mathbb{R}^2$ for any $\gamma \in [0, 1]$. Furthermore, if one takes a point, $x$, from the $\gamma$-level set of $A$ then one can only $y = x$ from the $\gamma$-level set of $B$ for the pair $(x, y)$ to belong to $[C]_\gamma$. This point-to-point interactivity relation is the strongest one that we can envisage between $\gamma$-level sets of marginal possibility distributions.

What can one say about the strength of interactivity between marginal distributions, $A(x) = 1 - x$ and $B(y) = 1 - y$, when their joint distribution, $F$, is defined, for example, by the Lukasiewicz $t$-norm? In this case

$$F(x, y) = \max\{A(x) + B(y) - 1, 0\} = \max\{1 - x + 1 - y - 1, 0\} = \max\{1 - x - y, 0\},$$

and $[F]_\gamma = \{(x, y) \mid x + y \leq 1 - \gamma\}$ is of symmetric triangular form for any $0 \leq \gamma < 1$. If we take, for example, $\gamma = 0.4$ then the pair $(0.3, 0.2)$ belongs to $[F]^{0.4}$ since $0.3 + 0.2 \leq 1 - 0.4$, but the pair $(0.4, 0.4)$ does not (see Fig. 1). In our approach we will define a uniform probability distribution on $[F]^{0.4}$ with marginal probability distributions denoted by $X_{0.4}$ and $Y_{0.4}$. The expected value of
this uniform probability distribution, \((0.2, 0.2)\), will be nothing else but the center of mass (or gravity) of the set \([F]^{0.4}\) of homogeneous density (for calculations see Section 4). Then the probabilistic correlation coefficient, denoted by \(\rho(X_{0.4}, Y_{0.4})\), will be negative since the ‘strength’ of pairs \((x, y)\) \(\in [F]^{0.4}\) that are discordant (i.e. \((x - 0.2)(y - 0.2) < 0\)) is bigger than the ‘strength’ of those ones that are concordant (i.e. \((x - 0.2)(y - 0.2) > 0\)). Then we define the index of interactivity as the weighted average of these correlation coefficients over the set of all membership grades.

Let \(A \in \mathcal{F}\) be fuzzy number with a \(\gamma\)-level set denoted by \([A]^{\gamma} = [a_1(\gamma), a_2(\gamma)]\), \(\gamma \in [0, 1]\) and let \(U_{\gamma}\) denote a uniform probability distribution on \([A]^{\gamma}\), \(\gamma \in [0, 1]\). Recall that the mean value of \(U_{\gamma}\) is \(M(U_{\gamma}) = (a_1(\gamma) + a_2(\gamma))/2\) and its variance is computed by \(\text{var}(U_{\gamma}) = (a_2(\gamma) - a_1(\gamma))^2)/12\).

### 2 Possibilistic mean value, variance, covariance and correlation

The \(f\)-weighted possibilistic mean value of a possibility distribution \(A \in \mathcal{F}\) is the \(f\)-weighted average of probabilistic mean values of the respective uniform distributions on the level sets of \(A\). That is, the \(f\)-weighted possibilistic mean value of \(A \in \mathcal{F}\), with \([A]^{\gamma} = [a_1(\gamma), a_2(\gamma)]\), \(\gamma \in [0, 1]\), is defined by [9],

\[
E_f(A) = \int_0^1 M(U_{\gamma})f(\gamma)d\gamma = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} f(\gamma)d\gamma, \quad (1)
\]

where \(U_{\gamma}\) is a uniform probability distribution on \([A]^{\gamma}\) for all \(\gamma \in [0, 1]\). This definition is based on Goetschel-Voxman ordering of fuzzy numbers [11], and it
can be considered as a particular case of the average index proposed by Campos and González in [1].

If $f(\gamma) \equiv 1$ the $f$-weighted possibilistic mean value coincides with the (i) generative expectation of fuzzy numbers introduced by Chanas and Nowakowski in ([3], page 47); (ii) middle-point-of-the-mean-interval defuzzication method proposed by Yager in ([23], page161).

**Remark 2.1** There exist several other ways to define mean values of fuzzy numbers, e.g. Dubois and Prade [4] defined an interval-valued expectation of fuzzy numbers, viewing them as consonant random sets. They also showed that this expectation remains additive in the sense of addition of fuzzy numbers. Using evaluation measures, Yoshida et al [24] introduced a possibility mean, a necessity mean and a credibility mean of fuzzy numbers that are different from (1). Surveying the results in quantitative possibility theory, Dubois [7] showed that some notions (e.g. cumulative distributions, mean values) in statistics can naturally be interpreted in the language of possibility theory.

The $f$-weighted possibilistic covariance between marginal possibility distributions of a joint possibility distribution is defined as the $f$-weighted average of probabilistic covariances between marginal probability distributions whose joint probability distribution is uniform on each level-set of the joint possibility distribution. That is, the $f$-weighted possibilistic covariance between $A, B \in \mathcal{F}$, (with respect to their joint distribution $C$), can be written as [10],

$$\text{Cov}_f(A, B) = \int_0^1 \text{cov}(X_{\gamma}, Y_{\gamma}) f(\gamma) d\gamma,$$

where $X_{\gamma}$ and $Y_{\gamma}$ are random variables whose joint distribution is uniform on $[C]^{\gamma}$ for all $\gamma \in [0, 1]$, and $\text{cov}(X_{\gamma}, Y_{\gamma})$ denotes their probabilistic covariance. It should be noted that the possibilistic covariance is an absolute measure of interactivity between marginal possibility distributions.

The measure of $f$-weighted possibilistic variance of $A$ is the $f$-weighted average of the probabilistic variances of the respective uniform distributions on the level sets of $A$. That is, the $f$-weighted possibilistic variance of $A$ is defined as [10]

$$\text{Var}_f(A) = \int_0^1 \text{var}(U_{\gamma}) f(\gamma) d\gamma = \int_0^1 \frac{(a_2(\gamma) - a_1(\gamma))^2}{12} f(\gamma) d\gamma.$$

There exist other approaches to define variance of fuzzy numbers, e.g. Dubois et al [6] defined the potential variance of a symmetric fuzzy interval by viewing it as a family of its $\alpha$-cut.
A measure of possibilistic correlation between marginal possibility distributions $A$ and $B$ of a joint possibility distribution $C$ has been defined in [2] as their possibilistic covariance divided by the square root of the product of their possibilistic variances. That is, the $f$-weighted measure of possibilistic correlation of $A, B \in F$ (with respect to their joint distribution $C$) is,

$$
\rho_{f}^{old}(A, B) = \frac{\text{Cov}_{f}(A, B)}{\sqrt{\text{Var}_{f}(A)} \sqrt{\text{Var}_{f}(B)}}
$$

where $U_{\gamma}$ is a uniform probability distribution on $[A]^{\gamma}$, and $V_{\gamma}$ is a uniform probability distribution on $[B]^{\gamma}$. Thus, the possibilistic correlation represents an average degree to which $X_{\gamma}$ and $Y_{\gamma}$ are linearly associated as compared to the dispersions of $U_{\gamma}$ and $V_{\gamma}$. We have the following result [2]. If $[C]^{\gamma}$ is convex for all $\gamma \in [0, 1]$ then $-1 \leq \rho_{f}^{old}(A, B) \leq 1$ for any $f$.

The presence of weighting function is not crucial in our theory: we can simple remove it from consideration by choosing $f(\gamma) \equiv 1$.

**Remark 2.2** There exist several other ways to define correlation coefficient for fuzzy numbers, e.g. Liu and Kao [16] used fuzzy measures to define a fuzzy correlation coefficient of fuzzy numbers and they formulated a pair of nonlinear programs to find the $\alpha$-cut of this fuzzy correlation coefficient, then, in a special case, Hong [12] showed an exact calculation formula for this fuzzy correlation coefficient. Vaidyanathan [21] introduced a new measure for the correlation coefficient between triangular fuzzy variables called credibilistic correlation coefficient.

### 3 An improved index of interactivity for fuzzy numbers

The main drawback of the definition of the former index of interactivity (2) is that it does not necessarily take its values from $[-1, 1]$ if some level-sets of the joint possibility distribution are not convex. For example, consider a joint possibility distribution defined by

$$
C(x, y) = 4x \cdot \chi_{T}(x, y) + 4/3(1 - x) \cdot \chi_{S}(x, y),
$$

where,

$$
T = \{(x, y) \in \mathbb{R}^{2} | 0 \leq x \leq 1/4, 0 \leq y \leq 1/4, x \leq y\},
$$

and

$$
S = \{(x, y) \in \mathbb{R}^{2} | 0 \leq x \leq 1/4, 0 \leq y \leq 1/4, x \geq y\}.
$$
and,

\[ S = \{(x, y) \in \mathbb{R}^2 \mid 1/4 \leq x \leq 1, 1/4 \leq y \leq 1, y \leq x \}. \]

Furthermore, we have,

\[ [C]^{\gamma} = \{(x, y) \in \mathbb{R}^2 \mid \gamma/4 \leq x \leq 1/4, x \leq y \leq 1/4\} \cup \{(x, y) \in \mathbb{R}^2 \mid 1/4 \leq x \leq 1 - 3/4\gamma, 1/4 \leq y \leq x\}. \]

We can see that \([C]^{\gamma}\) is not a convex set for any \(\gamma \in [0, 1)\) (see Fig. 2).

Then the marginal possibility distributions of (3) are computed by (see Fig. 3),

\[
A(x) = B(x) = \begin{cases} 
4x, & \text{if } 0 \leq x \leq 1/4 \\
4/3(1 - x), & \text{if } 1/4 \leq x \leq 1 \\
0, & \text{otherwise}
\end{cases}
\]

After some computations we get \(\rho_f^{old}(A, B) \approx 1.562\) for the weighting function \(f(\gamma) = 2\gamma\). We get here a value bigger than one since the variance of the first marginal distributions, \(X_\gamma\), exceeds the variance of the uniform distribution on the same support.

Let us now introduce a new index of interactivity between marginal distributions \(A\) and \(B\) of a joint possibility distribution \(C\) as the \(f\)-weighted average of the probabilistic correlation coefficients between the marginal probability distributions of a uniform probability distribution on \([C]^{\gamma}\) for all \(\gamma \in [0, 1]\). That is,

**Definition 3.1** The \(f\)-weighted index of interactivity of \(A, B \in F\) (with respect to
their joint distribution $C$) is defined by

$$
\rho_f(A, B) = \int_0^1 \rho(X_{\gamma}, Y_{\gamma}) f(\gamma) d\gamma \tag{4}
$$

where

$$
\rho(X_{\gamma}, Y_{\gamma}) = \frac{\text{cov}(X_{\gamma}, Y_{\gamma})}{\sqrt{\text{var}(X_{\gamma})} \sqrt{\text{var}(Y_{\gamma})}}
$$

and, where $X_{\gamma}$ and $Y_{\gamma}$ are random variables whose joint distribution is uniform on $[C]^\gamma$ for all $\gamma \in [0, 1]$.

In other words, the ($f$-weighted) index of interactivity is nothing else, but the $f$-weighted average of the probabilistic correlation coefficients $\rho(X_{\gamma}, Y_{\gamma})$ for all $\gamma \in [0, 1]$. It is clear that for any joint possibility distribution this new correlation coefficient always takes its value from interval $[-1, 1]$, since $\rho(X_{\gamma}, Y_{\gamma}) \in [-1, 1]$ for any $\gamma \in [0, 1]$ and $\int_0^1 f(\gamma) d\gamma = 1$. As for the joint possibility distribution defined by (3) we get $\rho_f(A, B) \approx 0.786$ for any $f$. Since $\rho_f(A, B)$ measures an average index of interactivity between the level sets of $A$ and $B$, we sometimes will call this measure as the $f$-weighted possibilistic correlation coefficient.

4 An example

Consider the case, when $A(x) = B(x) = (1 - x) \cdot \chi_{[0, 1]}(x)$, for $x \in \mathbb{R}$, that is $[A]^\gamma = [B]^\gamma = [0, 1 - \gamma]$, for $\gamma \in [0, 1]$. Suppose that their joint possibility distribution is given by $F(x, y) = (1 - x - y) \cdot \chi_T(x, y)$, where

$$
T = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x + y \leq 1\}.
$$
A $\gamma$-level set of $F$ is computed by

$$[F]^\gamma = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x + y \leq 1 - \gamma\}.$$ 

This situation is depicted on Fig. 4, where we have shifted the fuzzy sets to get a better view of the situation.

The density function of a uniform distribution on $[F]^\gamma$ can be written as

$$f(x, y) = \begin{cases} 
\frac{1}{\int_{[F]^\gamma} dxdy}, & \text{if } (x, y) \in [F]^\gamma \\
0, & \text{otherwise} 
\end{cases} = \begin{cases} 
\frac{2}{(1-\gamma)^2}, & \text{if } (x, y) \in [F]^\gamma \\
0, & \text{otherwise} 
\end{cases}$$

The marginal functions are obtained as

$$f_1(x) = \begin{cases} 
\frac{2(1-\gamma-x)}{(1-\gamma)^2}, & \text{if } 0 \leq x \leq 1 - \gamma \\
0, & \text{otherwise} 
\end{cases}$$

$$f_2(y) = \begin{cases} 
\frac{2(1-\gamma-y)}{(1-\gamma)^2}, & \text{if } 0 \leq y \leq 1 - \gamma \\
0, & \text{otherwise} 
\end{cases}$$

We can calculate the probabilistic expected values of the random variables $X_\gamma$ and $Y_\gamma$, whose joint distribution is uniform on $[F]^\gamma$ for all $\gamma \in [0, 1]$:

$$M(X_\gamma) = \frac{2}{(1-\gamma)^2} \int_0^{1-\gamma} x(1-\gamma-x) dx = \frac{1-\gamma}{3}.$$
and,

\[ M(Y_\gamma) = \frac{2}{(1 - \gamma)^2} \int_0^{1-\gamma} y(1 - \gamma - y) dy = \frac{1 - \gamma}{3}. \]

We calculate the variations of \( X_\gamma \) and \( Y_\gamma \) with the formula \( \text{var}(X) = M(X^2) - M(X)^2 \):

\[ M(X_\gamma^2) = \frac{2}{(1 - \gamma)^2} \int_0^{1-\gamma} x^2(1 - \gamma - x) dx = \frac{(1 - \gamma)^2}{6} \]

and,

\[ \text{var}(X_\gamma) = M(X_\gamma^2) - M(X_\gamma)^2 = \frac{(1 - \gamma)^2}{6} - \frac{(1 - \gamma)^2}{9} = \frac{(1 - \gamma)^2}{18}. \]

And similarly we obtain

\[ \text{var}(Y_\gamma) = \frac{(1 - \gamma)^2}{18}. \]

Using that

\[ M(X_\gamma Y_\gamma) = \frac{2}{(1 - \gamma)^2} \int_0^{1-\gamma} \int_0^{1-\gamma-x} xydydx = \frac{(1 - \gamma)^2}{12}, \]

\[ \text{cov}(X_\gamma, Y_\gamma) = M(X_\gamma Y_\gamma) - M(X_\gamma)M(Y_\gamma) = -\frac{(1 - \gamma)^2}{36}, \]

we can calculate the probabilistic correlation of the random variables:

\[ \rho(X_\gamma, Y_\gamma) = \frac{\text{cov}(X_\gamma, Y_\gamma)}{\sqrt{\text{var}(X_\gamma)} \sqrt{\text{var}(Y_\gamma)}} = -\frac{1}{2}. \]

And finally the \( f \)-weighted possibilistic correlation of \( A \) and \( B \):

\[ \rho_f(A, B) = \int_0^1 -\frac{1}{2} f(\gamma) d\gamma = -\frac{1}{2}. \]

We note here that using the former definition (2) we would obtain \( \rho_f^{old}(A, B) = -1/3 \) for the correlation coefficient (see [2] for details).

### 5 Some important examples

In this section we will show three important examples for the possibilistic correlation coefficient.
5.1 Non-interactive fuzzy numbers

If $A$ and $B$ are non-interactive then their joint possibility distribution is defined by $C = A \times B$. Since all $[C]^{\gamma}$ are rectangular and the probability distribution on $[C]^{\gamma}$ is defined to be uniform we get $\text{cov}(X^{\gamma}, Y^{\gamma}) = 0$, for all $\gamma \in [0, 1]$. So $\text{Cov}_f(A, B) = 0$ and $\rho_f(A, B) = 0$ for any weighting function $f$.

5.2 Perfect correlation

Fuzzy numbers $A$ and $B$ are said to be in perfect correlation, if there exist $q, r \in \mathbb{R}$, $q \neq 0$ such that their joint possibility distribution is defined by

$$C(x_1, x_2) = A(x_1) \cdot \chi_{\{qx_1 + r = x_2\}}(x_1, x_2) = B(x_2) \cdot \chi_{\{qx_1 + r = x_2\}}(x_1, x_2),$$

(5)

where $\chi_{\{qx_1 + r = x_2\}}$ stands for the characteristic function of the line

$$\{(x_1, x_2) \in \mathbb{R}^2 | qx_1 + r = x_2\}.$$

In this case we have

$$[C]^{\gamma} = \{(x, qx + r) \in \mathbb{R}^2 | x = (1 - t)a_1(\gamma) + ta_2(\gamma), t \in [0, 1]\}$$

where $[A]^{\gamma} = [a_1(\gamma), a_2(\gamma)]$; and $[B]^{\gamma} = q[A]^{\gamma} + r$, for any $\gamma \in [0, 1]$, and, finally,

$$B(x) = A\left(\frac{x - r}{q}\right),$$

for all $x \in \mathbb{R}$. Furthermore, $A$ and $B$ are in a perfect positive (negative) correlation if $q$ is positive (negative) in (5).

If $A$ and $B$ have a perfect positive (negative) correlation then from $\rho(X^{\gamma}, Y^{\gamma}) = 1$ ($\rho(X^{\gamma}, Y^{\gamma}) = -1$) [see [2] for details], for all $\gamma \in [0, 1]$, we get $\rho_f(A, B) = 1$ ($\rho_f(A, B) = -1$) for any weighting function $f$.

5.3 Mere shadows

Suppose that the joint possibility distribution of $A$ and $B$ is defined by,

$$C(x, y) = \begin{cases} A(x) & \text{if } y = 0 \\ B(y) & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$
Suppose further that,

\[ A(x) = B(x) = (1 - x) \cdot \chi_{[0,1]}(x), \]

for \( x \in \mathbb{R} \). Then a \( \gamma \)-level set of \( C \) is computed by

\[ [C]^{\gamma} = \{ (x, 0) \in \mathbb{R}^2 \mid 0 \leq x \leq 1 - \gamma \} \bigcup \{ (0, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1 - \gamma \}. \]

Since all \( \gamma \)-level sets of \( C \) are degenerated, i.e. their integrals vanish, we calculate everything as a limit of integrals. We calculate all the quantities with the \( \gamma \)-level sets:

\[ [C]^{\gamma}_\delta = \{ (x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1 - \gamma, 0 \leq y \leq \delta \} \bigcup \{ (x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1 - \gamma, 0 \leq x \leq \delta \}. \]

First we calculate the expected value and variance of \( X_\gamma \) and \( Y_\gamma \):

\[ M(X_\gamma) = \lim_{\delta \to 0} \frac{1}{\int_{[C]^{\gamma}_\delta} \, dx \, dy} \int_{[C]^{\gamma}_\delta} x \, dx = \frac{1 - \gamma}{4}, \]

\[ M(X^2_\gamma) = \lim_{\delta \to 0} \frac{1}{\int_{[C]^{\gamma}_\delta} \, dx \, dy} \int_{[C]^{\gamma}_\delta} x^2 \, dx = \frac{(1 - \gamma)^2}{6}, \]

\[ \text{var}(X_\gamma) = M(X^2_\gamma) - M(X_\gamma)^2 = \frac{(1 - \gamma)^2}{6} - \left( \frac{1 - \gamma}{4} \right)^2 = \frac{5(1 - \gamma)^2}{48}. \]
Because of the symmetry, the results are the same for $Y_\gamma$. We need to calculate their covariance,

$$M(X_\gamma Y_\gamma) = \lim_{\delta \to 0} \frac{1}{\int [C]_\delta^2} \int [C]_\delta^2 xydydx = 0,$$

Using this we obtain,

$$\text{cov}(X_\gamma, Y_\gamma) = -\frac{(1 - \gamma)^2}{16},$$

and for the correlation,

$$\rho(X_\gamma, Y_\gamma) = \frac{\text{cov}(X_\gamma, Y_\gamma)}{\sqrt{\text{var}(X_\gamma)}\sqrt{\text{var}(Y_\gamma)}} = -\frac{3}{5}.$$

Finally we obtain the $f$-weighted possibilistic correlation:

$$\rho_f(A, B) = \int_0^1 -\frac{3}{5} f(\gamma) d\gamma = -\frac{3}{5}.$$

In this extremal case, the joint distribution is unequivocally constructed from the knowledge that $C(x, y) = 0$ for positive $x, y$. Now we explain the reason for this negative correlation. Let us choose, for example, $\gamma = 0.4$ (see Fig. 7). The center of mass of $[C]_{0.4}$ is $(0.15, 0.15)$. The crucial point here is that if we choose any point, $x$, from $[A]_{0.4}$ then the only possible choice from $[B]_{0.4}$ can be $y = 0$, which is always less than 0.15, independently of the choice of $x$. In $[C]_{0.4}$ the strength of discordant points is much bigger than the strength of concordant points, with respect to the reference point $(0.15, 0.15)$.

6 Question

It is our guess that for these non-symmetrical, but identical marginal distributions, $A(x) = B(x) = (1 - x)$, for all $x \in [0, 1]$, one can not define any joint possibility distribution and any $f$ for which $\rho_f(A, B)$ could go below the value of $-3/5$. A possibility distribution $A$ is said to be symmetric if there exists a point $a \in \mathbb{R}$ such that $A(a - x) = A(a + x)$ for all $x \in \mathbb{R}$. If the membership functions of two symmetrical marginal possibility distributions are equal then we can easily define a joint possibility distribution in such a way that their possibilistic correlation coefficient will be minus one (see Subsection 5.2). And here comes our question: What is the lower limit for $f$-weighted possibilistic correlation coefficient between non-symmetrical marginal possibility distributions with the same membership function?
7 Conclusions

We have introduced a novel measure of (relative) index of interactivity between marginal distributions $A$ and $B$ of a joint possibility distribution $C$. The starting point of our approach is to equip the $\gamma$-level set of the joint possibility distribution with a uniform probability distribution. Then the correlation coefficient between its marginal probability distributions is considered to be an index of interactivity between the $\gamma$-level sets of $A$ and $B$. If $[C]^{\gamma}$ is rectangular for $0 \leq \gamma < 1$ then $A$ and $B$ are non-interactive and their index of interactivity is equal to zero. In the general case we have used the probabilistic correlation coefficient to measure the interactivity between the $\gamma$-level sets of $A$ and $B$, which, loosely speaking, measures the 'strength' of concordant points as to the 'strength' of discordant points of $[C]^{\gamma}$ with respect to the center of mass of $[C]^{\gamma}$. This new index of interactivity is meaningful for any joint possibility distribution.

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References


9 Follow-ups

In 2011 Harmati has answered the question of Section 6 and proved that the correlation coefficient depends only on the joint possibility distribution, but not directly on its marginal distributions. In other words exact knowledge of the marginal possibility distributions does not give any restrictions for the possibilistic correlation coefficient and also for the $f$-weighted possibilistic correlation coefficient. (see István Á. Harmati, A note on $f$-weighted possibilistic correlation for identical marginal possibility distributions, Fuzzy Sets and Systems, 165(2011) 106-110. doi: 10.1016/j.fss.2010.11.005).