Fuzzy reasoning for solving fuzzy mathematical programming problems *

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Abstract

We interpret fuzzy linear programming (FLP) problems with fuzzy coefficients and fuzzy inequality relations as multiple fuzzy reasoning schemes (MFR), where the antecedents of the scheme correspond to the constraints of the FLP problem and the fact of the scheme is the objective of the FLP problem. Then the solution process consists of two steps: first, for every decision variable \( x \in \mathbb{R}^n \), we compute the (fuzzy) value of the objective function, \( \text{MAX}(x) \), via sup-min convolution of the antecedents/constraints and the fact/objective, then an (optimal) solution to FLP problem is any point which produces a maximal element of the set \( \{ \text{MAX}(x) \mid x \in \mathbb{R}^n \} \) (in the sense of the given inequality relation). We show that our solution process for a classical (crisp) LP problem results in a solution in the classical sense, and (under well-chosen inequality relations and objective function) coincides with those suggested by Buckley [2], Delgado et al. [5, 6], Negoita [9], Ramik and Rimanek [12], Verdegay [16, 17] and Zimmermann [24]. Furthermore, we show how to extend the proposed solution principle to non-linear programming problems with fuzzy coefficients. We illustrate our approach by some simple examples.

Keywords: Compositional rule of inference, multiple fuzzy reasoning, fuzzy mathematical programming, possibilistic mathematical programming.

1 Statement of LP problems with fuzzy coefficients

We consider LP problems, in which all of the coefficients are fuzzy quantities (i.e. fuzzy sets of the real line \( \mathbb{R} \)), of the form

\[
\begin{align*}
\text{maximize} & \quad \tilde{c}_1 x_1 + \cdots + \tilde{c}_n x_n \\
\text{subject to} & \quad \tilde{a}_{ij} x_1 + \cdots + \tilde{a}_{in} x_n \lesssim \tilde{b}_i, \quad i = 1, \ldots, m, \tag{1}
\end{align*}
\]

where \( x \in \mathbb{R}^n \) is the vector of decision variables, \( \tilde{a}_{ij} \), \( \tilde{b}_i \) and \( \tilde{c}_j \) are fuzzy quantities, the operations addition and multiplication by a real number of fuzzy quantities are defined by Zadeh’s extension principle [20], the inequality relation \( \lesssim \) for constraints is given by a certain fuzzy relation and the objective function is to be maximized in the sense of a given crisp inequality relation \( \leq \) between fuzzy quantities.

The FLP problem (1) can be stated as follows: Find \( x^* \in \mathbb{R}^n \) such that

\[
\begin{align*}
\tilde{c}_1 x_1 + \cdots + \tilde{c}_n x_n & \quad \leq \quad \tilde{c}_1 x_1^* + \cdots + \tilde{c}_n x_n^* \\
\tilde{a}_{ij} x_1 + \cdots + \tilde{a}_{in} x_n & \quad \lesssim \quad \tilde{b}_i, \quad i = 1, \ldots, m,
\end{align*}
\]

i.e. we search for an alternative, \( x^* \), which maximizes the objective function subject to constraints.

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2 Preliminaries

In this section we set up the notations and recall some fuzzy inference rules needed for the proposed solution principle. A fuzzy set \( \tilde{a} \) of the real line \( \mathbb{R} \) is called a fuzzy quantity. We denote the family of fuzzy quantities by \( \mathcal{F} \). A fuzzy number \( [7] \) \( \tilde{a} \) is a fuzzy quantity with a normalized (i.e. there exists a unique \( a \in \mathbb{R} \), called peak of \( \tilde{a} \), such that \( \mu_{\tilde{a}}(a) = 1 \)), continuous, fuzzy convex membership function of bounded support.

A symmetric triangular fuzzy number \( \tilde{a} \) denoted by \( (a, \alpha) \) is defined as \( \mu_{\tilde{a}}(t) = 1 - |a-t|/\alpha \) if \( |a-t| \leq \alpha \) and \( \mu_{\tilde{a}}(t) = 0 \) otherwise, where \( a \in \mathbb{R} \) is the peak and \( 2\alpha > 0 \) is the spread of \( \tilde{a} \).

In the following \( \tilde{a} \) denotes the characteristic function of the singleton \( a \in \mathbb{R} \), i.e. \( \mu_{\tilde{a}}(t) = 1 \) if \( t = a \) and \( \mu_{\tilde{a}}(t) = 0 \) otherwise.

Let \( X \) be a non-empty set. The empty fuzzy set in \( X \), denoted by \( \emptyset \), is defined as \( \mu_{\emptyset}(x) = 0, \forall x \in X \).

A binary fuzzy relation \( W \) in \( X \) is a fuzzy subset of the Cartesian product \( X \times X \) and defined by its membership function \( \mu_W \). If \( \mu_W(u,v) \in \{0,1\}, \forall u,v \in X \) then \( W \) is called a crisp relation in \( X \).

Throughout this paper we shall use the terms relation and inequality relation interchangeably, i.e. we do not require any additional property for the later. However, we can get unexpected solutions if we use unjustifiable inequality relations to compare fuzzy quantities.

Let \( \leq \) be a crisp inequality relation in \( \mathcal{F} \). Then for all pairs \( \tilde{a}, \tilde{b} \in \mathcal{F} \) it induces a crisp binary relation in \( \mathbb{R} \) defined by

\[
\mu_{\tilde{a} \leq \tilde{b}}(u,v) = \begin{cases} 1 & \text{if } u = v, \text{ and } \tilde{a} \text{ and } \tilde{b} \text{ are in relation } \leq, \\ 0 & \text{otherwise.} \end{cases}
\]

It is clear that \( \mu_{\tilde{a} \leq \tilde{b}} = \emptyset \iff \tilde{a} \text{ and } \tilde{b} \text{ are not in relation } \leq. \)

If the inequality relation \( \leq \) is modeled by a fuzzy implication operator \( I \) then for all pairs \( \tilde{a}, \tilde{b} \in \mathcal{F} \) it induces a fuzzy binary relation in \( \mathbb{R} \) defined by

\[
\mu_{\tilde{a} \leq \tilde{b}}(u,v) = I(\mu_{\tilde{a}}(u), \mu_{\tilde{b}}(v)),
\]

e.g. if \( \leq \) is given by the Gödel implication operator then we have

\[
\mu_{\tilde{a} \leq \tilde{b}}(u,v) = \begin{cases} 1 & \text{if } \mu_{\tilde{a}}(u) \leq \mu_{\tilde{b}}(v), \\ \mu_{\tilde{b}}(v) & \text{otherwise.} \end{cases}
\]

(2)

If an inequality relation \( \leq \) in \( \mathcal{F} \) is not crisp then instead of \( \leq \) we will usually write \( \preceq \).

We will use the following crisp inequality relations in \( \mathcal{F} \):

\[
\tilde{a} \leq \tilde{b} \iff \max\{\tilde{a}, \tilde{b}\} = \tilde{b}
\]

(3)

where \( \max \) is the ordinary extension of the two-placed function, \( \max \), and defined by Zadeh’s extension principle,

\[
\tilde{a} \leq \tilde{b} \iff \tilde{a} \subseteq \tilde{b},
\]

(4)

where \( \tilde{a} \subseteq \tilde{b} \) if \( \mu_{\tilde{a}}(u) \leq \mu_{\tilde{b}}(u), \forall u \in \mathbb{R} \),

\[
\tilde{a} \leq \tilde{b} \iff \text{peak}(\tilde{a}) \leq \text{peak}(\tilde{b})
\]

(5)

where \( \tilde{a} \) and \( \tilde{b} \) are fuzzy numbers, and \( \text{peak}(\tilde{a}) \) and \( \text{peak}(\tilde{b}) \) denote their peaks,

\[
\tilde{a} \leq \tilde{b} \iff a \leq b,
\]

(6)
where $\bar{a}$ and $\bar{b}$ are fuzzy singletons. Let $\Gamma$ be an index set, $\bar{a}_\gamma \in F, \gamma \in \Gamma$, and let $\preceq$ be a crisp inequality relation in $F$. We say that $\bar{a}$ is a maximal element of the set

$$G := \{\bar{a}_\gamma | \gamma \in \Gamma\}$$

(7)

if $\bar{a}_\gamma \preceq \bar{a}$ for all $\gamma \in \Gamma$ and $\bar{a} \in G$. A fuzzy quantity $\bar{A}$ is called an upper bound of $G$ if $\bar{a}_\gamma \preceq \bar{A}$ for all $\gamma \in \Gamma$. A fuzzy quantity $\bar{A}$ is called a least upper bound (supremum) of $G$ if it is an upper bound and if there exists an upper bound $\bar{B}$, such that $\bar{B} \preceq \bar{A}$, then $\bar{A} \preceq \bar{B}$. If $\bar{A}$ is a least upper bound of $G$, then we write

$$\bar{A} = \sup\{\bar{a}_\gamma | \gamma \in \Gamma\}$$

It is easy to see that, depending on the definition of the inequality relation, the set (7) may have many maximal elements (suprema) or the set of maximal elements (suprema) may be empty. For example, (i) if $\{\operatorname{peak}(\bar{a}) | \gamma \in \Gamma\}$ is a bounded and closed subset of the real line then $G$ has at least one maximal element in the sense of relation (5); (ii) $G$ always has a unique supremum in relation (4), but usually does not have maximal elements; (iii) if there exists $u \in \mathbb{R}$, such that $\bar{a}_\gamma(v) = 0$, for all $v \geq u$ and $\gamma \in \Gamma$ then $G$ has infinitely many suprema in relation (3).

The degree of possibility of the statement "$\bar{a}$ is smaller or equal to $\bar{b}$", which we write $\operatorname{Poss}[\bar{a} \preceq \bar{b}]$, is defined by

$$\operatorname{Poss}[\bar{a} \preceq \bar{b}] = \sup_{x \leq y} \min\{\mu_{\bar{a}}(x), \mu_{\bar{b}}(y)\},$$

which induces the following relation in $\mathbb{R}$

$$\mu_{\bar{a} \preceq \bar{b}}(u,v) = \begin{cases} \operatorname{Poss}[\bar{a} \preceq \bar{b}] & \text{if } u = v \\ 0 & \text{otherwise} \end{cases}$$

(8)

We shall use the compositional rule of inference scheme with several relations (called Multiple Fuzzy Reasoning Scheme) [19] which has the general form

$$\begin{array}{ll}
\text{Fact} & X \text{ has property } P \\
\text{Relation 1:} & X \text{ and } Y \text{ are in relation } W_1 \\
\text{...} & \text{...} \\
\text{Relation m:} & X \text{ and } Y \text{ are in relation } W_m \\
\text{Consequence:} & Y \text{ has property } Q \\
\end{array}$$

where $X$ and $Y$ are linguistic variables taking their values from fuzzy sets in classical sets $U$ and $V$, respectively, $P$ and $Q$ are unary fuzzy predicates in $U$ and $V$, respectively, $W_i$ is a binary fuzzy relation in $U \times V$, $i=1,\ldots,m$. The consequence $Q$ is determined by [19]

$$Q = P \circ \bigcap_{i=1}^m W_i$$

or in detail,

$$\mu_Q(y) = \sup_{x \in U} \min\{\mu_P(x), \mu_{W_1(x,y)}, \ldots, \mu_{W_m(x,y)}\}.$$
3 Multiply fuzzy reasoning for solving FLP problems

We consider FLP problems as MFR schemes, where the antecedents of the scheme correspond to the constraints of the FLP problem and the fact of the scheme is interpreted as the objective function of the FLP problem.

Then the solution process consists of two steps: first, for every decision variable \( x \in \mathbb{R}^n \), we compute the value of the objective function, \( MAX(x) \), via sup-min convolution of the antecedents/constraints and the fact/objective, then an (optimal) solution to the FLP problem is any point which produces a maximal element of the set \( \{ MAX(x) \mid x \in \mathbb{R}^n \} \) (in the sense of the given inequality relation).

We interpret the FLP problem (1) as MFR schemes of the form

\[
\begin{align*}
\text{Antecedent 1} & \quad \text{Constraint}_1(x) := \tilde{a}_{11}x_1 + \cdots + \tilde{a}_{1n}x_n \preceq \tilde{b}_1 \\
\vdots \quad \text{Antecedent m} & \quad \text{Constraint}_m(x) := \tilde{a}_{m1}x_1 + \cdots + \tilde{a}_{mn}x_n \preceq \tilde{b}_m \\
\text{Fact} & \quad \text{Goal}(x) := \tilde{c}_1x_1 + \cdots + \tilde{c}_nx_n \\
\text{Consequence} & \quad MAX(x)
\end{align*}
\]

where \( x \in \mathbb{R}^n \) and the consequence (i.e. the value of the objective function subject to constraints at \( x \)) \( MAX(x) \) is computed as follows

\[
MAX(x) = \text{Goal}(x) \circ \bigcap_{i=1}^{m} \text{Constraint}_i(x),
\]

i.e.

\[
\mu_{MAX(x)}(v) = \sup_u \min \{ \mu_{\text{Goal}(x)}(u), \mu_{\text{Constraint}_1(x)}(u, v), \ldots, \mu_{\text{Constraint}_m(x)}(u, v) \}. \tag{9}
\]

Then an optimal value of the objective function of problem (1), denoted by \( M \), is defined as

\[
M := \sup \{ MAX(x) \mid x \in \mathbb{R}^n \}, \tag{10}
\]

where \( \sup \) is understood in the sense of the given crisp inequality relation for the objective function. Finally, a solution \( x^* \in \mathbb{R}^n \) to problem (1) is obtained by solving the equation

\[
MAX(x) = M.
\]

The set of solutions of problem (1) is non-empty iff the set of maximizing elements of

\[
\{ MAX(x) \mid x \in \mathbb{R}^n \} \tag{11}
\]

is non-empty.

Remark 3.1 Apart from the deterministic LP, \( \max \{ \langle c, x \rangle \mid Ax \leq b \} \), where we simply compute the value of the objective function as \( c_1y_1 + \cdots + c_ny_n \) at any feasible point \( y \in \mathbb{R}^n \) and do not care about non-feasible points, in FLP problem (1) we have to consider the whole decision space, because every point \( y \) from \( \mathbb{R}^n \) has a (fuzzy) degree of feasibility (given by the fuzzy relations \( \text{Constraint}_i(y), i = 1, \ldots, m \)). We have right to compute the value of the objective function of (1) at \( y \in \mathbb{R}^n \) as \( \tilde{c}_1y_1 + \cdots + \tilde{c}_nx_n \) if there are no constraints at all (if there are no rules in a fuzzy reasoning scheme then the consequence takes the value of the observation automatically).

Remark 3.2 To determine a maximal element of the set (11) even in a crisp inequality relation is usually a very complicated process. However, this problem can lead to a crisp LP problem (see Zimmermann
4 Extension to FMP problems with fuzzy coefficients

In this section we show how the proposed approach can be extended to non-linear FMP problems with fuzzy coefficients. Generalizing the classical MP problem

\[
\begin{align*}
\text{maximize} & \quad g(c, x) \\
\text{subject to} & \quad f_i(a_i, x) \leq b_i, \ i = 1, \ldots, m,
\end{align*}
\]

where \( x \in \mathbb{R}^n \), \( c = (c_1, \ldots, c_k) \) and \( a_i = (a_{i1}, \ldots, a_{il}) \) are vectors of crisp coefficients, we consider the following FMP problem

\[
\begin{align*}
\text{maximize} & \quad g(\tilde{c}_1, \ldots, \tilde{c}_k, x) \\
\text{subject to} & \quad f_i(\tilde{a}_{i1}, \ldots, \tilde{a}_{il}, x) \preceq \tilde{b}_i, \ i = 1, \ldots, m,
\end{align*}
\]

where \( x \in \mathbb{R}^n \), \( \tilde{c}_h, h = 1, \ldots, k \), \( \tilde{a}_{is}, s = 1, \ldots, l \), and \( \tilde{b}_i \) are fuzzy quantities, the functions \( g(\tilde{c}, x) \) and \( f_i(\tilde{a}, x) \) are defined by Zadeh’s extension principle, and the inequality relation \( \preceq \) is defined by a certain fuzzy relation. We interpret the above FMP problem as MFR schemes of the form

\[
\begin{align*}
\text{Antecedent 1:} & \quad \text{Constraint}_1(x) := f_i(\tilde{a}_{i1}, \ldots, \tilde{a}_{il}, x) \preceq \tilde{b}_i \\
& \quad \ldots \\
\text{Antecedent m:} & \quad \text{Constraint}_m(x) := f_m(\tilde{a}_{m1}, \ldots, \tilde{a}_{ml}, x) \preceq \tilde{b}_m \\
\text{Fact:} & \quad \text{Goal}(x) := g(\tilde{c}_1, \ldots, \tilde{c}_k, x) \\
\text{Consequence} & \quad \text{MAX}(x)
\end{align*}
\]

Then the solution process is carried out analogously to the linear case, i.e an optimal value of the objective function, \( M \), is defined by (10), and a solution \( x^* \in \mathbb{R}^n \) is obtained by solving the equation \( \text{MAX}(x) = M \).

5 Relation to classical LP problems

In this section we show that our solution process for classical LP problems results in a solution in the classical sense. A classical LP problem can be stated as follows

\[
\max \quad < c, x > \\
\text{subject to} \quad Ax \leq b
\]

Let \( X^* \) be the set of solutions and if \( X^* \neq \emptyset \) then let \( v^* = < c, x^* > \) denote the optimal value of the objective function of (12). An element \( x \) from \( \mathbb{R}^n \) is said to be feasible if it satisfies the inequality \( Ax \leq b \).
Generalizing the crisp LP problem (12) we consider the FLP problem (1) with fuzzy singletons and crisp inequality relations (6)

\[ \begin{align*}
\text{maximize} & \quad \text{Goal}(x) := \bar{c}_1 x_1 + \cdots + \bar{c}_k x_n \\
\text{subject to} & \quad \text{Constraint}_1(x) := \bar{a}_{11} x_1 + \cdots + \bar{a}_{1n} x_n \leq \bar{b}_i \\
& \quad \ldots \\
& \quad \text{Constraint}_m(x) := \bar{a}_{m1} x_1 + \cdots + \bar{a}_{mn} x_n \leq \bar{b}_m 
\end{align*} \]

(13)

where \( \bar{a}_{ij}, \bar{b}_i \) and \( \bar{c}_j \) denote the characteristic function of the crisp coefficients \( a_{ij}, b_i \) and \( c_j \), respectively, and the inequality relation \( \leq \) is defined by

\[ \bar{a}_{i1} x_1 + \cdots + \bar{a}_{in} x_n \leq \bar{b}_i \text{ iff } a_{i1} x_1 + \cdots + a_{in} x_n \leq b_i, \]

i.e.

\[ \mu_{\bar{a}_{i1} x_1 + \cdots + \bar{a}_{in} x_n \leq \bar{b}_i}(u, v) = \begin{cases} 
1 & \text{if } u = v \text{ and } <a_i, x> \leq b_i \\
0 & \text{otherwise} \end{cases} \]

(14)

Then from (9) we get

\[ \mu_{\text{MAX}}(x)(v) = \begin{cases} 
1 & \text{if } v = <c, x> \text{ and } Ax \leq b \\
0 & \text{otherwise}, \end{cases} \]

which can be written in the form

\[ \text{MAX}(x) = \begin{cases} <c, x> & \text{if } x \text{ is feasible} \\
0 & \text{otherwise}, \end{cases} \]

consequently, if \( x \) and \( x' \) are feasible then

\[ \text{MAX}(x) \leq \text{MAX}(x') \quad \text{iff} \quad <c, x> \leq <c, x'>, \]

and if \( x' \) is feasible, but \( x'' \) is not feasible then

\[ \text{MAX}(x'') \leq \text{MAX}(x'), \]

since \( \text{MAX}(x'') \) is empty. Therefore from (10) we get \( M = \bar{v}^* \), and \( x^* \) satisfies the equality \( \text{MAX}(x^*) = M \) if and only if \( v^* = <c, x^*> \), i.e. \( x^* \in X^* \).

This means that LP problem (12) and FLP problem (13) have the same solution-set, and the optimal value of the FLP problem is the characteristic function of the optimal value of the LP problem.

6 Crisp objective function and fuzzy coefficients in constraints

FLP problems with crisp inequality relations in fuzzy constraints and crisp objective function can be formulated as follows (see Negoita’s robust programming [9], Ramik and Rimanek [12], and Werners [18])

\[ \begin{align*}
\max & \quad <c, x> \\
\text{subject to} & \quad \bar{a}_{i1} x_1 + \cdots + \bar{a}_{in} x_n \leq \bar{b}_i, \quad i = 1, \ldots, m.
\end{align*} \]

(15)

It is easy to see that problem (15) is equivalent to the crisp MP problem

\[ \max_{x \in X} <c, x> \]

(16)
where,
\[ X = \bigcap_{i=1}^{m} X_i = \bigcap_{i=1}^{m} \{ x \in \mathbb{R}^n \mid \tilde{a}_{i1}x_1 + \cdots + \tilde{a}_{in}x_n \leq \tilde{b}_i \} . \]

Now we show that our approach leads to the same crisp MP problem (15). Consider problem (1) with fuzzy singletons in the objective function
\[
\begin{align*}
\text{max} & \quad < \tilde{c}, x > \\
\text{subject to} & \quad \tilde{a}_{i1}x_1 + \cdots + \tilde{a}_{in}x_n \leq \tilde{b}_i, \quad i = 1, \ldots, m. \\
\end{align*}
\]
where the inequality relation \( \leq \) is defined by
\[
\mu_{\tilde{a}_{i1}x_1 + \cdots + \tilde{a}_{in}x_n \leq \tilde{b}_i}(u, v) = \begin{cases} 1 & \text{if } u = v \text{ and } x \in X_i \\ 0 & \text{otherwise} \end{cases}
\]
Then we have
\[
\mu_{\text{MAX}(x)}(v) = \begin{cases} 1 & \text{if } v = < c, x > \text{ and } x \in X \\ 0 & \text{otherwise} \end{cases}
\]
Thus, to find a maximizing element of the set \( \{ \text{MAX}(x) \mid x \in \mathbb{R}^n \} \) in the sense of the given inequality relation we have to solve the crisp problem (16).

7 Fuzzy objective function and crisp constraints

Consider the FLP problem (1) with fuzzy coefficients in the objective function and fuzzy singletons in the constraints
\[
\begin{align*}
\text{max} & \quad \tilde{c}_1x_1 + \cdots + \tilde{c}_nx_n \\
\text{subject to} & \quad \tilde{a}_{i1}x_1 + \cdots + \tilde{a}_{in}x_n \leq \tilde{b}_i, \quad i = 1, \ldots, m, \\
\end{align*}
\]
where the inequality relation \( \leq \) is defined by (14) and the objective function is to be maximized in relation (3), i.e.
\[
\text{MAX}(x') \leq \text{MAX}(x'') \iff \max\{\text{MAX}(x'), \text{MAX}(x'')\} = \text{MAX}(x'').
\]
Then \( \mu_{\text{MAX}(x)}(v), \forall v \in \mathbb{R} \), is the optimal value of the following crisp MP problem
\[
\begin{align*}
\text{maximize} & \quad \mu_{\tilde{c}_1x_1 + \cdots + \tilde{c}_nx_n}(v) \\
\text{subject to} & \quad Ax \leq b,
\end{align*}
\]
and the problem of computing a solution to FLP problem (17) leads to the same crisp multiple objective parametric linear programming problem obtained by Delgado et al. [5, 6] and Verdegay [16, 17].

8 Relation to Zimmermann’s soft constraints

Consider Zimmermann’s LP with crisp coefficients and soft constraints: Find \( x \) such that
\[
\begin{align*}
< c, x > & \preceq z \\
< a_i, x > & \preceq b_i, \quad i = 1, \ldots, m, \\
\end{align*}
\]
where the inequality relation \( \preceq \) is defined by
\[
< a_i, x > \preceq b_i = \begin{cases} 1 & \text{if } a_i, x \leq b_i \\ 1 - (b_i - < a_i, x >)/d_i & \text{if } b_i \leq < a_i, x > \leq b_i + d_i \\ 0 & \text{otherwise} \end{cases}
\]

for $i = 1, \ldots, m$, and
\[
< c, x > \preceq z = \begin{cases} 
1 & \text{if } < c, x > \leq z \\
1 - (z - < c, x >)/d_0 & \text{if } z \leq < c, x > \leq z + d_0 \\
0 & \text{otherwise}
\end{cases} 
\] (20)

An optimal solution $x^*$ to (18) is determined from the crisp LP
\[
\begin{align*}
\text{maximize} & \quad \lambda \\
\text{subject to} & \quad 1 - (z - < c, x >)/d_0 \geq \lambda \\
& \quad 1 - (b_i - < a_i, x >)/d_i \geq \lambda, \quad i = 1, \ldots, m, \\
& \quad 0 \leq \lambda \leq 1.
\end{align*}
\] (21)

The following theorem can be proved directly by using the definitions (4) and (10).

**Theorem 8.1** The FLP problem
\[
\begin{align*}
\text{maximize} & \quad 1 \\
\text{subject to} & \quad \bar{c}_1 x_1 + \cdots + \bar{c}_n x_n \preceq \bar{z} \\
& \quad \bar{a}_{i1} x_1 + \cdots + \bar{a}_{in} x_n \preceq \bar{b}_i, \quad i = 1, \ldots, m.
\end{align*}
\] (22)

where $\mu_1(u) = 1, \forall u \in \mathbb{R}$, the objective function is to be maximized in relation (4) and $\preceq$ is defined by (19) and (20), i.e.
\[
\mu_{<a_i,x>} \preceq_{b_i} (u,v) = \begin{cases} 
1 & \text{if } < a_i, x > \leq b_i \\
1 - (b_i - < a_i, x >)/d_i & \text{if } b_i \leq < a_i, x > \leq b_i + d_i \\
0 & \text{otherwise}
\end{cases}
\]
for $i = 1, \ldots, m$, and
\[
\mu_{<c,x>} \preceq_{z} (u,v) = \begin{cases} 
1 & \text{if } < c, x > \leq z \\
1 - (z - < c, x >)/d_0 & \text{if } z \leq < c, x > \leq z + d_0 \\
0 & \text{otherwise}
\end{cases}
\]
has the same solution-set as problem (21).

9 Relation to Buckley’s possibilistic LP

We show that when the inequality relations in an FLP problem are defined in a possibilistic sense then the optimal value of the objective function is equal to the possibility distribution of the objective function defined by Buckley [1].
Consider a possibilistic LP
\[
\begin{align*}
\text{maximize} & \quad Z := \bar{c}_1 x_1 + \cdots + \bar{c}_n x_n \\
\text{subject to} & \quad \bar{a}_{i1} x_1 + \cdots + \bar{a}_{in} x_n \leq \bar{b}_i, \quad i = 1, \ldots, m.
\end{align*}
\] (23)

The possibility distribution of the objective function $Z$, denoted by $\text{Poss}[Z = z]$, is defined by [1]
\[
\text{Poss}[Z = z] = \sup_x \min \{ \text{Poss}[Z = z \mid x], \text{Poss}[< \bar{a}_1, x > \leq \bar{b}_1], \ldots, \text{Poss}[< \bar{a}_m, x > \leq \bar{b}_m] \},
\]
where \( \text{Poss}[Z = z \mid x] \), the conditional possibility that \( Z = z \) given \( x \), is defined by

\[
\text{Poss}[Z = z \mid x] = \mu_{\tilde{c}_1 x_1 + \tilde{c}_n x_n}(z).
\]

The following theorem can be proved directly by using the definitions of \( \text{Poss}[Z = z] \) and \( \mu_M(v) \).

**Theorem 9.1** For the FLP problem

\[
\begin{align*}
\text{maximize} & \quad \tilde{c}_1 x_1 + \cdots + \tilde{c}_n x_n \\
\text{subject to} & \quad \tilde{a}_i x_1 + \cdots + \tilde{a}_m x_n \leq \tilde{b}_i, \ i = 1, \ldots, m.
\end{align*}
\]

(24)

where the inequality relation \( \leq \) is defined by (8) and the objective function is to be maximized in relation (4), i.e.

\[
\text{MAX}(x') \leq \text{MAX}(x'') \quad \text{iff} \quad \text{MAX}(x') \subseteq \text{MAX}(x''),
\]

the following equality holds

\[
\mu_M(v) = \text{Poss}[Z = v], \forall v \in \mathbb{R}^n.
\]

So, if the inequality relations for constraints are defined in a possibilistic sense and the objective function is to be maximized in relation (4) then the optimal value of the objective function of FLP problem (24) is equal to the possibility distribution of the objective function of possibilistic LP (23).

### 10 Examples

We illustrate our approach by two simple FMP problems.

**Example 1** Consider the FLP problem

\[
\begin{align*}
\text{maximize} & \quad \tilde{c} x \\
\text{subject to} & \quad \tilde{a} \preceq \tilde{a}, \ 0 \leq x \leq 4,
\end{align*}
\]

(25)

where \( \tilde{c} = (1, 1) \) is a fuzzy number of symmetric triangular form, \( \tilde{a} \) is a fuzzy number with membership function

\[
\mu_{\tilde{a}}(u) = \begin{cases} 
1 - u/4 & \text{if } 0 \leq x \leq 4 \\
0 & \text{otherwise},
\end{cases}
\]

the inequality relation for the constraint is defined by (2)

\[
\mu_{\tilde{a} \preceq \tilde{a}}(u, v) = \begin{cases} 
1 & \text{if } \mu_{\tilde{a}}(u) \leq \mu_{\tilde{a}}(v) \\
\mu_{\tilde{a}}(v) & \text{otherwise}
\end{cases}
\]

and the inequality relation for the objective function is given by (4). Then the corresponding fuzzy reasoning scheme is

\[
\begin{array}{c|c}
\text{Antecedent} & \tilde{a} \preceq \tilde{a} \\
\text{Fact} & \tilde{c} x \\
\text{Consequence} & \text{MAX}(x)
\end{array}
\]
Figure 2: The membership function of MAX(2.5).

It is easy to compute that for \(0 \leq x \leq 2\) (see Fig.1.)

\[
\mu_{\text{MAX}}(x)(v) = \begin{cases} 
1 & \text{if } 0 \leq v \leq x \\
(4 - v)/(4 - x) & \text{if } x < v \leq 4 \\
0 & \text{otherwise}
\end{cases}
\]

and for \(2 \leq x \leq 4\) (see Fig.2.)

\[
\mu_{\text{MAX}}(x)(v) = \begin{cases} 
1 & \text{if } 0 \leq v \leq x \\
(1/x - 4/x^2)v + 4/x & \text{if } x < v \leq 2x \\
2 - 4/x & \text{otherwise}
\end{cases}
\]

So, if \(0 \leq x' \leq x'' \leq 4\) then from (4) we get

\[
\text{MAX}(x') \leq \text{MAX}(x'') \leq \text{MAX}(4) = 1
\]

This means that \(x^* = 4\) is the unique solution and \(1\) is the optimal value of (25).

It differs from the defuzzified case

\[
\begin{aligned}
\text{maximize} & \quad x \\
\text{subject to} & \quad 0 \leq 0, \quad 0 \leq x \leq 4
\end{aligned}
\]

where the coefficients are the peaks of the fuzzy coefficients of FLP problem (25), because the solution \(x^* = 4\) of the crisp problem is equal to the solution of (25), but the optimal value of the FLP problem is too large \(\mu_M(v) = 1, \forall v \in \mathbb{R}\) (because the Gödel implication enlarges \(\text{MAX}(x), \forall x \in \mathbb{R}\) by taking into account all membership values \(\mu_{\tilde{a}}(u)\) and \(\mu_{\tilde{a}}(v)\) separately).

**Example 2** Consider the FMP problem

\[
\begin{aligned}
\text{maximize} & \quad \tilde{c}x \\
\text{subject to} & \quad (\tilde{a}x)^2 \preceq \tilde{b}, \quad x \geq 0
\end{aligned}
\]

where \(\tilde{a} = (2,1), \quad \tilde{b} = (1,1)\) and \(\tilde{c} = (3,1)\) are fuzzy numbers of symmetric triangular form, the inequality relation \(\preceq\) is defined in a possibilistic sense, i.e.

\[
\mu_{\tilde{a}x \preceq \tilde{b}}(u,v) = \begin{cases} 
\text{Poss}[\tilde{a}x \leq \tilde{b}] & \text{if } u = v, \\
0 & \text{otherwise}
\end{cases}
\]

and the inequality relation for the values of the objective function is defined by (5) (with the difference that subnormal values of the objective function are considered smaller than normal ones), i.e.

\[
\text{MAX}(x') \leq \text{MAX}(x'') \quad \text{iff} \quad \text{peak}(\text{MAX}(x')) \leq \text{peak}(\text{MAX}(x''))
\]

where \(\text{MAX}(x')\) and \(\text{MAX}(x'')\) are fuzzy numbers, and \(\text{MAX}(x') \leq \text{MAX}(x'')\) if \(\text{MAX}(x')\) is subnormal fuzzy quantity and \(\text{MAX}(x'')\) is a fuzzy number.

It is easy to compute that

\[
\text{Poss}[(\tilde{a}x)^2 \preceq \tilde{b}] = \begin{cases} 
1 & \text{if } x \leq 1/2, \\
\theta(x) & \text{if } 1/2 \leq x \leq \sqrt{2} \\
0 & \text{if } x \geq \sqrt{2}
\end{cases}
\]
where
\[
\theta(x) = \frac{-1 - 2x^2 + \sqrt{1 + 12x^2}}{2x^2}.
\]
and \(MAX(x)\) is a fuzzy number if \(0 \leq x \leq 1/2\).

Therefore, the unique solution to FMP problem (26) is \(x^* = 1/2\) and the optimal value of the objective function is
\[
\mu_{\text{MAX}}(x)(v) = \mu_{\text{MAX}}(1/2)(v) = \begin{cases} 
4 - 2v & \text{if } 3/2 \leq v \leq 2, \\
2v - 2 & \text{if } 1 \leq v \leq 3/2 \\
0 & \text{otherwise}
\end{cases}
\]
So, the optimal solution to FMP problem (26) is equal to the optimal solution of crisp problem
\[
\begin{align*}
\text{maximize} & \quad 3x \\
\text{subject to} & \quad (2x)^2 \leq 1, \ x \geq 0,
\end{align*}
\]
where the coefficients are the peaks of the fuzzy coefficients of problem (26), and the optimal value of problem (26), which can be called "\(v\) is approximately equal to 3/2" (see Fig. 3.), can be considered as an approximation of the optimal value of the crisp problem \(v^* = 3/2\).

11 Concluding remarks

We have interpreted FLP problems with fuzzy coefficients and fuzzy inequality relations as MFR schemes and shown a method for finding an optimal value of the objective function and an optimal solution. In the general case the computerized implementation of the proposed solution principle is not easy. To compute \(MAX(x)\) we have to solve a generally non-convex and non-differentiable mathematical programming problem. However, the stability property of the consequence in MFR schemes under small changes of the membership function of the antecedents [8] guarantees that small rounding errors of digital computation and small errors of measurement in membership functions of the coefficients of the FLP problem can cause only a small deviation in the membership function of the consequence, \(MAX(x)\), i.e. every successive approximation method can be applied to the computation of the linguistic approximation of the exact \(MAX(x)\). As was pointed out in Section 3, to find an optimal value of the objective function, \(M\), from the equation \(MAX(x) = M\) can be a very complicated process (for related works see [3, 4, 9, 10, 16, 17, 22, 23]) and very often we have to put up with a compromise solution [13]. An efficient fuzzy-reasoning-based method is needed for the exact computation of \(M\). Our solution principle can be applied to multiple criteria mathematical programming problems with fuzzy coefficients. This topic will be the subject of future research.

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13 Follow ups

The results of this paper have been improved and/or generalized in the following works.

**in journals**

http://dx.doi.org/10.1057/palgrave.jors.2602609

The use of fuzzy numbers leads us to represent the problem as one of fuzzy mathematical programming. There have been works related to the properties and solution methodologies for solving fuzzy linear programming. Zimmermann (1976, 1978) proposed a method for linear programming problems. Similar methods were proposed in Rommelfanger (1996), Gasimov and Yenilmez (2002) and Fullér and Zimmermann (1993). We formulate the problem as a fuzzy integer linear program. The fuzzy problem is then transformed to a multi-objective problem with crisp numbers. The multiple objectives deal with the most likely of the fuzzy numbers as well as their spread in either direction. This multi-objective problem is then solved using a method proposed by Young-Jou and Ching-Lai (1994) that attempts to maximize the worst case satisfaction level of all the objectives. (page 718)

A21-c27 Sylvia Encheva, Sharil Tumin, Problem Identification Based on Fuzzy Functions, WSEAS TRANSACTIONS ON ADVANCES IN ENGINEERING EDUCATION, Issue 4, Volume 6, April 2009, pp. 111-120. 2009


http://dx.doi.org/10.1007/s00500-007-0207-6

http://dx.doi.org/10.1016/j.amc.2005.11.161

Fang and Hu [5] consider linear programming with fuzzy constraint coefficients (see also [A21]). (page 206)

http://dx.doi.org/10.1080/03052150600603165

http://dx.doi.org/10.1016/j.fss.2004.06.008

Computational methods that use fuzzy logic have attracted attention in recent years; most of them have addressed optimization problems. The methods of [1,2] were for solving a crisp linear programming problem by searching and reasoning. Extension of these methods to fuzzy mathematical programs in which the linear program has fuzzy coefficients was accomplished in [3,4,A21]. (page 600)

http://dx.doi.org/10.1109/TSMCB.2002.804361

http://dx.doi.org/10.1016/S0031-3203(98)00133-2

http://dx.doi.org/10.1016/S0165-0114(96)00259-X
The use of fuzzy set provides imprecise class membership information and is widely applied in diverse areas such as control, cluster analysis, decision making, engineering systems, etc. See, for example [2,3, A21, . . .]. (page 171)

http://dx.doi.org/10.1142/S0218488597000221

A21-c15 B. Müller, Short-term planning of the production program in dairies by considering the subjective vagueness of data in mathematical models on the basis of fuzzy sets, Kieler Milchwirtschaftliche Forschungsberichte: Veroeffentlichungen der Bundesanstalt fuer Milchforschung, 47(4), pp. 307-337. 1995

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http://dx.doi.org/10.1007/978-90-481-3656-8_34

http://dx.doi.org/10.1109/ICSMC.2009.5345943

Zimmermann in [4] and [A21] presented the most used method to solve this problem. (page 263)


http://dx.doi.org/10.1007/978-3-642-02937-0_18
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16