On Irregularities of Bidegreed Graphs

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Abstract: A graph is regular if all its vertices have the same degree. Otherwise a graph is irregular. To measure how irregular a graph is, several graph topological indices were proposed including: the Collatz-Sinogowitz index [8], the variance of the vertex degrees [7], the irregularity of a graph [4], and recently proposed the total irregularity of a graph [1]. Here, we compare the above mentioned irregularity measures for bidegreed graphs.

Keywords: topological graph indices; complete split graph; 2-walk linear graph

1 Introduction

All graphs considered here are simple and undirected. Let G be a graph of order $n = |V(G)|$ and size $m = |E(G)|$. For $v \in V(G)$, the degree of $v$, denoted by $d_G(v)$, is the number of edges incident to $v$. The adjacency matrix $A(G)$ of a graph $G$ is a matrix with rows and columns labeled by graph vertices, with a 1 or a 0 in position $(v_i, v_j)$ according to whether vertices $v_i$ and $v_j$ are adjacent or not. The characteristic polynomial $\phi(G, t)$ of $G$ is defined as characteristic polynomial of $A(G)$: $\phi(G, \lambda) = \det(\lambda I_n - A(G))$, where $I_n$ is $n \times n$ identity matrix. The set of eigenvalues of the adjacent matrix $A(G)$ of a graph $G$ is called a graph spectrum. The largest eigenvalue of $A(G)$, denoted by $\rho(G)$, is called the spectral radius of $G$. An eigenvalue of a graph $G$ is called main eigenvalue if it has an eigenvector the sum of whose entries is not equal to zero.

In the sequel, we present the irregularity measures consider in this paper. Collatz-Sinogowitz [8] introduced the irregularity measure of a graph $G$ as

$$CS(G) = \rho(G) - \frac{2m}{n}. \quad (1)$$
An alternative to CS(G) is the variance of the vertex degrees
\[
\text{Var}(G) = \frac{1}{n} \sum_{i=1}^{n} d_G^2(v_i) - \left( \frac{1}{n^2} \sum_{i=1}^{n} d_G(v_i) \right)^2.
\] (2)

Bell [7] was first who has compared CS(G) and Var(G) and showed that they are not always compatible. Albertson [4] defines the imbalance of an edge \(e = uv \in E\) as \(|d_G(u) - d_G(v)|\) and the irregularity of \(G\) as
\[
\text{irr}(G) = \sum_{uv \in E} |d_G(u) - d_G(v)|.
\] (3)

Recently, in [1] a new measure of irregularity of a simple, undirected graph, so-called the total irregularity, was defined as
\[
\text{irr}_t(G) = \frac{1}{2} \sum_{u,v \in V(G)} |d_G(u) - d_G(v)|.
\] (4)

More about the above presented irregularity measures, comparison studies of them, and other attempts to measure the irregularity of a graph, one can find in [3,6,10–12]. It is interesting that the above four irregularity measures are not always compatible for some pairs of graphs. In this paper we study the relations between the above mentioned irregularity measures for bidegreed graphs.

A universal vertex is the vertex adjacent to all other vertices. A set of vertices is said to be independent when the vertices are pairwise non-adjacent. The vertices from an independent set are independent vertices.

The degree set, denoted by \(D(G)\), of a simple graph \(G\) is the set consisting of the distinct degrees of vertices in \(G\).

The distance between two vertices in a graph is the number of edges in a shortest path connecting them. The eccentricity of a vertex \(v\) in a connected graph \(G\) is the maximum graph distance between \(v\) and any other vertex of \(G\). The radius of a graph \(G\), denoted by \(\text{rad}(G)\), is the minimum graph eccentricity of any graph vertex of \(G\). The diameter of a graph \(G\), denoted by \(\text{diam}(G)\), is the maximal graph eccentricity of any graph vertex of \(G\).

Let \(m_{r,s}\) denotes the number of edges in \(G\) with end-vertex degrees \(r\) and \(s\), and let \(n_r\) denotes the numbers of vertices \(n\) \(G\) with degree \(r\). Numbers \(m_{r,s}\) and \(n_r\) are referred as the edge-parameters and the vertex-parameters of \(G\), respectively. The mean degree of a graph \(G\) is defined as \(\bar{d}(G) = 2m/n\). Graphs \(G_1\) and \(G_2\) are said to be edge-equivalent if for their corresponding edge-parameters sets \(\{m_{r,s}(G_1) > 0\} = \{m_{r,s}(G_2) > 0\}\) holds. Analogously, they are called vertex-equivalent if for their vertex-parameters sets \(\{n_r(G_1) > 0\} = \{n_r(G_2) > 0\}\) is fulfilled. It is easy to see that if two graphs are edge-equivalent, then they are vertex-equivalent, as well.
For two graphs \(G_1\) and \(G_2\) with disjoint vertex sets \(V(G_1)\) and \(V(G_2)\) and disjoint edge sets \(E(G_1)\) and \(E(G_2)\) the disjoint union of \(G_1\) and \(G_2\) is the graph \(G = G_1 \cup G_2\) with the vertex set \(V(G_1) \cup V(G_2)\) and the edge set \(E(G_1) \cup E(G_2)\).

The join \(G + H\) of simple undirected graphs \(G\) and \(H\) is the graph with the vertex set \(V(G + H) = V(G) \cup V(H)\) and the edge set \(E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}\). Let \(C_n\) denote a cycle on \(n\) vertices. Further, let \(K_n\) denote the complete graph on \(n\) vertices, and \(tK_1\) denote the graph with \(t\) isolated vertices and no edges.

A graph \(G\) is a complete \(k\)-partite graph if there is a partition \(V_1 \cup \cdots \cup V_k = V(G)\) of the vertex set, such that \(uv \in E(G)\) if and only if \(u\) and \(v\) are in different parts of the partition. A connected bipartite graph \(G\) is semiregular if every edge of \(G\) joins a vertex of degree \(\delta\) to a vertex of degree \(\Delta\).

A connected graph \(G\) is called a balanced irregular graph if the equality \(\text{irr}(G) = \text{irr}_r(G)\) holds.

The rest of the paper is structured as follows. In Section 2 we present some types of bidegreed graphs and some known results related to the above mentioned irregularity measures. In Section 3 we investigate new relations between irregularity indices of bidegreed graphs. Bidegreed graphs with same irregularity indices are investigated in Section 4. We conclude with final remarks and open problems in Section 5.

2 Some types of bidegreed graphs and known results

A graph \(G\) is called bidegreed if its degree set \(\mathcal{D}(G) = \{\Delta, \delta\}\) with \(\Delta > \delta \geq 1\). In the sequel, we present some special types of connected bidegreed graphs that will be of interest later.

i) A bidegreed graph is called a balanced bidegreed graph if the equality \(n_\Delta n_\delta = m_{\Delta, \delta}\) holds for it. It should be noted that the complete bipartite graphs, for which \(m_{\Delta, \delta} = n_\Delta n_\delta = \Delta \delta\) holds, form a subset of balanced bidegreed graphs.

ii) A balanced bidegreed graph with \(n\) vertices is called a complete split graph if it contains \(q = n_\Delta \geq 1\) universal vertices and \(n - q\) independent vertices [5]. Thus, a complete split graph, denoted by \(G_{cs}(n, q)\), can be obtained as join of \(n - q\) graphs \(K_1\) and the complete graph \(K_q\), i.e., \(G_{cs}(n, q) = (n - q)K_1 + K_q\). An existing complete split graph \(G_{cs}(n, q)\) is uniquely defined by their parameters \(n\) and \(q\). This implies that two complete split graphs with identical \(n, q\) parameters are isomorphic. For a complete split graph the equalities \(m = m_{\Delta, \delta} + m_{\Delta, \Delta}\) and \(2m = (2n - 1)\delta - \delta^2\) hold [5].

iii) A balanced bidegreed graph is called a complete split-like graph, denoted by \(G_{csl}(n, q, \delta)\), if it has \(q \geq 1\) universal vertices. This implies that for a complete split-like graph the equality \(qn_\delta = m_{\Delta, \delta}\) holds. The complete split
graphs represent a subset of complete split-like graphs. It is easy to see that if \( G \) is a complete split-like graph then the equalities \( \text{rad}(G) = 1 \) and \( \text{diam}(G) = 2 \) are fulfilled. In Fig. 1 non-isomorphic complete split-like graphs with 5 and 6 vertices are depicted. Note that they are not complete split graphs.

![Fig. 1. Complete split-like graphs](a) \( G_{csl}(5, 1, 2) \) and (b) \( G_{csl}(6, 2, 3) \)

Also note that since for a complete split-like graph \( G \) \( qn_δ = m_{Δ, δ} \), it follows that if \( G \) is not a complete bipartite graph, then \( G \) is non-bipartite and contains a triangle.

iv) In a particular case, if \( q = 1 \), then a complete split-like graph is called a **generalized windmill graph** and is denoted by \( G_{csl}(n, 1, δ) \). We would like to recall that the classical windmill graph, denoted by \( W_d(k, p) \), can be constructed by joining \( p \) copies of the complete graph \( K_k \) with a common vertex. For a generalized windmill graph the equality \( m = m_{Δ, δ} + m_{δ, δ} \) is fulfilled. It follows that the star graphs \( S_n \) with \( n \geq 3 \) vertices, the wheel graphs \( W_n \) with \( n \geq 5 \) vertices, and the classical windmill graphs \( W_d(k, p) \) with \( (k - 1)p + 1 \) vertices and \( pk(k - 1)/2 \) edges defined for \( k \geq 2 \) and \( p \geq 2 \) positive integers, form the subsets of generalized windmill graphs. In Fig. 2 two non-isomorphic generalized windmill graphs are depicted.

Next, we state some known results that will be used afterwords.

**Lemma 1** ([16]). Let \( G \) be a connected bidegreed graph with spectral radius \( ρ(G) \). Then

\[
ρ(G) = \sqrt{\frac{1}{n} \sum_{u \in V(G)} d^2(u)} = \sqrt{Δδ},
\]

if and only if \( G \) is a semiregular connected bipartite graph.

**Lemma 2** ([15]). Let \( G \) be a connected graph with mean degree \( \overline{d}(G) = 2m/n \), and just two main eigenvalues, \( ρ \) and \( μ < ρ \), where \( ρ \) is the spectral
radius of $G$. Then

$$\text{Var}(G) = \frac{1}{n} \sum_{u \in V(G)} d^2(u) - \left( \frac{2m}{n} \right)^2 = \left( \rho - \frac{2m}{n} \right) \left( \frac{2m}{n} - \mu \right).$$

**Lemma 3 ([15]).** Let $G$ be a connected graph with spectral radius $\rho$. Then $G$ is a semiregular bipartite graph if and only if the main eigenvalues of $G$ are $\rho$ and $-\rho$.

**Lemma 4 ([13]).** Let $G$ be a connected graph with spectral radius $\rho$. Then

$$\rho(G) \leq \frac{\delta - 1 + \sqrt{\delta^2 + 2(2m - \delta n)}}{2}.$$

Equality holds if and only if $G$ is regular or a bidegreed graph in which each vertex is of degree either $\delta$ or $n - 1$.

### 3 Relations between irregularity indices - new results

In this section, we present some new results about the relations between irregularity indices of bidegreed graphs. We start with the following simple proposition.

**Proposition 1.** Let $G(\Delta, \delta)$ be a connected bidegreed graph having $n_\Delta$ and $n_\delta$ vertices with degree $\Delta$ and $\delta$, respectively. Then the following relations hold:

1. $$m = m(G(\Delta, \delta)) = m_{\Delta,\Delta} + m_{\Delta,\delta} + m_{\delta,\delta} \geq m_{\Delta,\delta},$$
2. $$\text{irr}(G(\Delta, \delta)) = m_{\Delta,\delta}(\Delta - \delta),$$
3. $$\text{irr}_t(G(\Delta, \delta)) = n_\Delta n_\delta (\Delta - \delta) = n_\Delta (n - n_\Delta)(\Delta - \delta),$$
4. $$\text{irr}_t(G(\Delta, \delta)) = \frac{n_\Delta n_\delta}{m_{\Delta,\delta}} \text{irr}(G(\Delta, \delta)).$$

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Fig. 2. Two generalized windmill graphs
Proof. It is obvious that for a connected bidegreed graph \( G(\Delta, \delta) \) the equality \( m = m_{\Delta, \delta} \) holds if and only if \( G(\Delta, \delta) \) is semiregular. The equalities (6), (7) and (8) follow from the definitions of irregularity indices.

Because the function \( f(n_\Delta) = n - n_\Delta \) has a maximum value for \( n_\Delta = n/2 \), we have the following corollary.

**Corollary 1.** For a connected bidegreed graph \( G(\Delta, \delta) \) it holds that

\[
\text{irr}_t(G(\Delta, \delta)) = n_\Delta(n - n_\Delta)(\Delta - \delta) \leq \frac{n^2}{4}(\Delta - \delta).
\]  

(9)

Inequality (9) is sharp. There exist bidegreed graphs with \( n \) vertices for which \( \text{irr}_t(G(\Delta, \delta)) = \frac{n^2(\Delta - \delta)}{4} \). Such bidegreed graphs with 8 vertex and deegre set \( \{3, 4\} \) are shown in Fig. 3(a). These graphs are non edge-equivalent, but only vertex equivalent, and the equality \( n_3 = n_4 = n/2 = 4 \) holds for them. Another example of bidegreed graphs that satisfy equality in (9) is given in Fig. 3(b). Those graphs are with 8 vertices and have deegre set \( \{2, 3\} \). They are edge-equivalent, and satisfy the equality \( n_2 = n_3 = n/2 = 4 \). It is interesting to note that the graphs in Fig. 3(b) are not only edge-equivalent \( (m_{2,3} = 8, m_{3,3} = 2) \), but they have identical spectral radius \( (1 + \sqrt{17})/2 \), as well. Consequently, all considered irregularity indices (CS, Var, irr and irrt ) are identical for them.

![Fig. 3. Examples of non-isomorphic bidegreed graphs with 8 vertices with identical maximum total irregularity indices](image)

**Proposition 2.** Let \( G(\Delta, \delta) \) be a connected bidegreed graph, then

\[
\text{irr}_t(G(\Delta, \delta)) = \frac{\Delta - \delta}{\Delta \delta} \left( m^2 - (m_{\Delta, \Delta} - m_{\delta, \delta})^2 \right) \leq \frac{\Delta - \delta}{\Delta \delta} m^2.
\]

The equality holds if \( m_{\Delta, \Delta} = m_{\delta, \delta} \).
Proof. For any bidegreed graph $G(\Delta, \delta)$, it holds that
\[
\Delta n_{\Delta} = m_{\Delta, \delta} + 2m_{\Delta, \Delta}, \quad \text{and} \\
\delta n_{\Delta} = m_{\Delta, \delta} + 2m_{\delta, \delta}.
\]
This together with (7) implies that
\[
\text{irr}(G(\Delta, \delta)) = \Delta - \delta \Delta \delta \left( m_{\Delta, \delta} + 2m_{\Delta, \Delta} \right) \left( m_{\Delta, \delta} + 2m_{\delta, \delta} \right).
\]
Since $m_{\Delta, \delta} = m - m_{\Delta, \Delta} - m_{\delta, \delta}$, it follows that
\[
\text{irr}(G(\Delta, \delta)) = \frac{\Delta - \delta}{\Delta \delta} \left( m^2 - (m_{\Delta, \Delta} - m_{\delta, \delta})^2 \right) \leq \frac{\Delta - \delta}{\Delta \delta} m^2. \quad (10)
\]
The equality in (10) is obtained when $m_{\Delta, \Delta} = m_{\delta, \delta}$. This condition holds for the bidegreed graphs with 10 vertices and 12 edges in Fig. 4. Consequently all of them have the same maximum total irregularity index $\text{irr} = n_2n_3 = 6 \cdot 4 = 24$.

Fig. 4. Bidegreed graphs having identical vertex degree set ($n_3 = 4, n_2 = 6$) and identical maximum total irregularity index irr = 24

Among bidegreed graphs having identical vertex degree set ($n_{\Delta}, n_{\delta}$), the semiregular graphs (for which the equality $m_{\Delta, \Delta} = m_{\delta, \delta}=0$ holds) possess the maximal irregularity $\text{irr}(G)$, as it is a case with graphs $J_c$ and $J_d$ in Fig. 4.

4 Bidegreed graphs with same irregularity indices

In the following we will show that there exists a broad class of bidegreed graphs having “similar irregularity”, or in other words, there exist non-isomorphic graph pairs for which two (or more than two) irregularity indices are equal. Moreover, we will show that there are some particular classes of bidegreed graphs whose irregularity indices are considered algebraically dependent quantities.
4.1 Balanced bidegreed graphs

From the definition of balanced bidegreed graphs, it follows that

$$\text{irr}(G(\Delta, \delta)) = \text{irr}_t(G(\Delta, \delta)) = n_{\Delta n_\delta}(\Delta - \delta) = m_{\Delta, \delta}(\Delta - \delta).$$

This implies that the balanced bidegreed graphs form a subset of balanced irregular graphs.

**Proposition 3.** Let $G(\Delta, \delta)$ be a balanced bidegreed graph for which $m_{\Delta, \Delta} = 0$ or $m_{\delta, \delta} = 0$ hold. Then

$$\text{irr}(G(\Delta, \delta)) = \text{irr}_t(G(\Delta, \delta)) = (2m - \Delta \delta)(\Delta - \delta).$$

**Proof.** For any bidegreed graph $G(\Delta, \delta)$

$$\Delta n_\Delta = m_{\Delta, \delta} + 2m_{\Delta, \Delta},$$

$$\delta n_\delta = m_{\Delta, \delta} + 2m_{\delta, \delta}.$$  

Consequently, we get

$$n_{\Delta n_\delta} = m_{\Delta, \delta} = \frac{(m_{\Delta, \delta} + 2m_{\Delta, \Delta})(m_{\Delta, \delta} + 2m_{\delta, \delta})}{\Delta \delta},$$

and

$$m_{\Delta, \delta}^2 + (2(m_{\Delta, \Delta} + m_{\delta, \delta}) - \Delta \delta)m_{\Delta, \delta} + 4m_{\Delta, \Delta}m_{\delta, \delta} = 0.$$  

Taking into consideration that $m_{\Delta, \Delta} + m_{\delta, \delta} = m - m_{\Delta, \delta}$, we have

$$m_{\Delta, \delta}^2 + (\Delta \delta - 2m)m_{\Delta, \delta} - 4m_{\Delta, \Delta}m_{\delta, \delta} = 0.$$  

Because $m_{\Delta, \delta}$ is a positive number it is easy to see that the proper solution of the equation above is

$$n_{\Delta n_\delta} = m_{\Delta, \delta} = \frac{1}{2} \left( 2m - \Delta \delta + \sqrt{(2m - \Delta \delta)^2 + 16m_{\Delta, \Delta}m_{\delta, \delta}} \right).$$

If as a particular case the equality $m_{\Delta, \Delta}m_{\delta, \delta} = 0$ holds for graph $G(\Delta, \delta)$, one obtains

$$n_{\Delta n_\delta} = m_{\Delta, \delta} = 2m - \Delta \delta,$$

from which the main result follows.

**Example 1.** We present two infinite sequences of balanced bidegreed graphs with the property $m_{\Delta, \Delta}m_{\delta, \delta} = 0$. The first infinite sequence is comprised of graphs $B(k)$, where $k$ is a positive integer. The case $k = 2$ is depicted in Fig. 5(a). A graph $B(k)$ has a vertex degree distribution $n_3 = 2k$ and $n_{2k} = 2$, and edge number $m = 5k$, where $k \geq 2$ positive integer. It is easy to see that for graphs $B(k)$, the equality $m_{2k, 2k} = 0$ holds.
The second infinite sequence is comprised of $k$-gonal bipyramids. A $k$-gonal bipyramid, with integer $k \geq 3$, is formed by joining a $k$-gonal pyramid and its mirror image base-to-base. It is a polyhedron having $2k$ triangular faces. The case $k = 6$ is depicted in Fig. 5(b) and redrawn in Fig. 5(c) for a better illustration. The graph $P(k)$ of a $k$-gonal bipyramid belongs to the family of balanced bidegreed graphs with degree 4 and $k$. For these graphs the equalities $n_4 n_k = m_{4k} = 2k$, $m = 3k$ and $m_{k, k} = 0$ hold.

4.2 Complete split graphs and complete split-like graphs

Proposition 4 ([2]). There exist a complete split graph pairs with $n$ vertices $G_{cs}(n, q)$ and $G_{cs}(n, q+1)$ with certain $n$ and $q$ positive integers, for which the equality $\text{irr}_t(G_{cs}(n, q)) = \text{irr}_t(G_{cs}(n, q + 1)) = \text{irr}(G_{cs}(n, q)) = \text{irr}(G_{cs}(n, q + 1))$ holds.

Example 2. The smallest complete split graph pair with this property is the star graph on 5 vertices $G_{cs}(5, 1)$, and the graph $G_{cs}(5, 2)$ are depicted in Fig. 6.

For graphs $G_{cs}(5, 1)$ and $G_{cs}(5, 2)$ the following equality holds: $\text{irr}_t(G_{cs}(5, 1)) = \text{irr}_t(G_{cs}(5, 2)) = \text{irr}(G_{cs}(5, 1)) = \text{irr}(G_{cs}(5, 2)) = 12$.

Proposition 5. Let $G_{cs}(n, q, \delta)$ be a complete split-like graph. Then

$$\text{irr}(G_{cs}(n, q, \delta)) = \text{irr}_t(G_{cs}(n, q, \delta)) = q(n - q)(n - 1 - \delta).$$

Proof. Since the complete split-like graphs form a subset of balanced bidegreed graphs, it is easy to see that

$$\text{irr}(G_{csl}(n, q, \delta)) = m_{\Delta, \delta} |\Delta - \delta| = n_{\Delta} n_{\delta} |\Delta - \delta| = q(n - q)(n - 1 - \delta)$$

$=$ $\text{irr}_t(G_{csl}(n, q, \delta)).$
Proposition 6. There exist complete split-like graph pairs $G_{csl}(n, q, \delta)$ and $G_{csl}(n', q', \delta')$ with different $n, n', q, q', \delta$ and $\delta'$ parameters, for which the equality
\[
\text{irr}_1(G_{csl}(n, q, \delta)) = \text{irr}_1(G_{csl}(n', q', \delta')) = \text{irr}(G_{cs}(n, q, \delta)) = \text{irr}(G_{cs}(n', q', \delta')) = 8
\]
holds.

Proof. A complete split-like graph pair with this property is the graph pair $G_{csl}(5, 1, 2)$ and $G_{csl}(6, 2, 4)$ depicted in Fig. 7. For these graphs, equality
\[
\text{irr}_1(G_{csl}(5, 1, 2)) = \text{irr}_1(G_{csl}(6, 2, 4)) = \text{irr}(G_{cs}(5, 1, 2)) = \text{irr}(G_{cs}(6, 2, 4)) = 8
\]
holds.

There are several ways to construct complete split-like graphs. For example, a complete split-like graph with $n$ vertices $G_{cs}(n, q, \delta)$ can be generated using the following graph operations:
\[
G_{cs}(n, q, \delta) = K_q + \bigcup_{j=1}^{q} H(j, R)
\]
In the formula above, $K_q$ is the complete graph on $q \geq 1$ vertices, $H(j, R)$ are $R \geq 1$ regular connected graphs for $j = 1, 2, \ldots, J$.

As an example, in Fig. 8 two non-isomorphic edge-equivalent complete split-like graphs are shown. These complete split-like graphs are defined as $G_{csl}^1(14, 2, 4) = K_2 + C_{12}$ and $G_{csl}^2(14, 2, 4) = K_2 + (C_3 \cup C_4 \cup C_5)$, respectively. It is easy to see that $\text{irr}(G_{csl}^1(14, 2, 4)) = \text{irr}(G_{csl}^2(14, 2, 4)) = 216$.

From the previous considerations the following result follows.

**Proposition 7.** Let $G_1$ and $G_2$ be edge-equivalent complete split-like graphs. Then the equalities $\text{irr}(G_1) = \text{irr}(G_2) = \text{irr}(G_1) = \text{irr}(G_2)$, $\text{Var}(G_1) = \text{Var}(G_2)$ and $\text{CS}(G_1) = \text{CS}(G_2)$ are fulfilled for them.

**Proof.** Because $G_1$ and $G_2$ are edge-equivalent graphs, this implies that the equalities $\text{irr}(G_1) = \text{irr}(G_2), \text{irr}(G_1) = \text{irr}(G_2)$ and $\text{Var}(G_1) = \text{Var}(G_2)$ hold. Moreover, because $G_1$ and $G_2$ are complete split-like graphs, in which each vertex is of degree $\delta$ or $n-1$, it follows from Lemma 4 that their spectral radii are identical.

For an illustration of Proposition 7, see the complete split-like graph pair depicted in Fig. 8.

### 4.3 Semiregular graphs

It is important to note that except the complete bidegreed bipartite graphs, the semiregular graphs do not belong to the family of balanced bidegreed graphs.
Proposition 8. Let \( S_1(\Delta_1, \delta_1) \) and \( S_2(\Delta_2, \delta_2) \) be semiregular graphs for which \( \Delta = \Delta_1 = \Delta_2, \delta = \delta_1 = \delta_2, \) and \( m_{\Delta, \delta} = m(S_1) = m(S_2) \) hold. Then,

\[
CS(S_1) = CS(S_2) = \sqrt{\Delta \delta} - \frac{2 \Delta \delta}{\Delta + \delta},
\]

and

\[
\text{Var}(S_i) = \left( \sqrt{\Delta \delta} + \frac{2 \Delta \delta}{\Delta + \delta} \right) CS(S_i),
\]

for \( i = 1, 2, \) where \( CS(G) \) is the Collatz-Sinogowitz irregularity index of a graph \( G. \)

Proof. It is easy to see that for a semiregular graph \( S \) with \( n \) vertices

\[
n = n_\Delta + n_\delta = \frac{m_{\Delta, \delta}}{\Delta} + \frac{m_{\Delta, \delta}}{\delta} = \frac{\Delta + \delta}{\Delta \delta} m_{\Delta, \delta}.
\]

This implies that for the mean degrees \( \overline{d} \) we have

\[
\overline{d}(S_1) = \overline{d}(S_2) = \frac{2 m_{\Delta, \delta}}{n} = \frac{2 \Delta \delta}{\Delta + \delta}.
\]

From Lemma 1 one obtains

\[
\rho = \rho(S_1) = \rho(S_2) = \sqrt{\Delta \delta}.
\]

consequently, we have

\[
CS(S_1) = CS(S_2) = \rho - \frac{2 m_{\Delta, \delta}}{n} = \sqrt{\Delta \delta} - \frac{2 \Delta \delta}{\Delta + \delta}.
\]

Moreover, from Lemmas 2 and 3, it follows that for a semiregular graphs \( S \)

\[
\text{Var}(S) = \left( \rho - \frac{2 m}{n} \right) \left( \frac{2 m}{n} + \rho \right) = \rho^2 - \left( \frac{2 m}{n} \right)^2 = \Delta \delta - \left( \frac{2 \Delta \delta}{\Delta + \delta} \right)^2
\]

\[
= \left( \sqrt{\Delta \delta} + \frac{2 \Delta \delta}{\Delta + \delta} \right) CS(S).
\]

This implies that

\[
\text{Var}(S_i) = \left( \sqrt{\Delta \delta} + \frac{2 \Delta \delta}{\Delta + \delta} \right) CS(S_i),
\]

for \( i = 1, 2. \)

Proposition 9. Let \( S_1(\Delta_1, \delta_1) \) and \( S_2(\Delta_2, \delta_2) \) be semiregular graphs for which \( \Delta = \Delta_1 = \Delta_2, \delta = \delta_1 = \delta_2, \) and \( m_{\Delta, \delta} = m(S_1) = m(S_2) \) hold. Then, the equalities \( \text{irr}(S_1) = \text{irr}(S_2), \text{irr}(S_1) = \text{irr}(S_2) \) are fulfilled for them.
Proof. It is obvious that
\[ \text{irr}(S_1) = \text{irr}(S_2) = m_{\Delta,\delta}(\Delta - \delta). \]
Moreover, because for a semiregular graph \( n_{\Delta}n_{\delta} = \frac{m^2_{\Delta,\delta}}{\Delta\delta} \), we get
\[ \text{irr}_t(S_1) = \text{irr}_t(S_2) = n_{\Delta}n_{\delta}(\Delta - \delta) = \frac{\Delta - \delta}{\Delta\delta}m^2_{\Delta,\delta}. \]

As a consequence of Proposition 8 and 9, we have the following result.

**Corollary 2.** Let \( S_1(\Delta_1,\delta_1) \) and \( S_2(\Delta_2,\delta_2) \) be semiregular graphs for which \( \Delta = \Delta_1 = \Delta_2, \delta = \delta_1 = \delta_2, \) and \( m_{\Delta,\delta} = m(S_1) = m(S_2) \) hold. Then the equalities \( \text{irr}_t(S_1) = \text{irr}_t(S_2), \text{irr}(S_1) = \text{irr}(S_2), \text{Var}(S_1) = \text{Var}(S_2) \) and \( \text{CS}(S_1) = \text{CS}(S_2) \) are fulfilled for them.

Graphs \( J_c \) and \( J_d \) depicted in Fig. 4 satisfy Corollary 2. From Proposition 9, we have the following corollary.

**Corollary 3.** Let \( S(\Delta,\delta) \) be a semiregular graph. Then,
\[ \text{irr}_t(S(\Delta,\delta)) = \frac{\text{irr}^2(S(\Delta,\delta))}{\Delta\delta(\Delta - \delta)}. \]

### 4.4 Bidegreed graphs with identical CS, Var, irr and irr\(_t\) indices

In Fig. 3(b), Proposition 7 and Corollary 2 examples of pairs of bidegreed graphs were presented, with the property that both graphs from a given pair have identical CS, Var, irr and irr\(_t\). Next, we present another such pair of graphs. A 6-vertex graph pair with degree set \( \{2,3\} \) and with identical CS, Var, irr and irr\(_t\) indices is depicted in Fig. 9. These graphs are edge-equivalent \( (m_{2,3} = 4, m_{3,3} = 4) \), and they have identical spectral radius \( 1 + \sqrt{3} \).

In the sequel, we show that there exists an infinitely large family of pairs of bidegreed graphs with identical CS, Var, irr and irr\(_t\) indices. For that purpose, first we need the following definition:

Let \( d_2(v) \) denote the sum of the degrees of all vertices adjacent to a vertex \( v \) in a graph \( G \). Then, \( G \) is called 2-walk linear (more precisely, 2-walk \((a,b)\)-linear) if there exists a unique rational numbers pair \( (a,b) \) such that
\[ d_2(v) = a \cdot d(v) + b \]
holds for every vertex \( v \) of \( G \).
Lemma 5 ([14]). A graph $G$ has exactly two main eigenvalues if and only if $G$ is 2-walk linear. Moreover, if $G$ is a 2-walk $(a,b)$-linear connected graph, then parameters $a$ and $b$ must be integers, and the spectral radius of $G$ is

$$\rho = \frac{1}{2} \left( a + \sqrt{a^2 + 4b} \right).$$

Using the above lemma we will demonstrate by examples that there are infinitely many bidegreed graph pairs having identical irregularity indices $\CS, \Var, \irr, \irrt$.

**Example 3.** Consider the two infinite sequences of bidegreed graphs denoted by $G_a(k)$ and $G_b(k)$ (an illustration when $k = 5$ is given in Fig. 10). Both $G_a(k)$ and $G_b(k)$ are of order $3k$, where $k \geq 3$. Graphs $G_a(k)$ and $G_b(k)$ are edge-equivalent, because the identities $m_{2,2} = k, m_{2,4} = 2k, m_{4,4} = k, m = 4k$ are fulfilled. Moreover, $G_a(k)$ and $G_b(k)$ are 2-walk $(3,0)$ linear graphs. By Lemma 5, it follows that they have identical spectral radius which is equal to 3. It is easy to show that for graphs $G_a(k)$ and $G_b(k)$ the following equalities hold: $\CS(G_a(k)) = \CS(G_b(k)) = 1/3, \Var(G_a(k)) = \Var(G_b(k)) = 8/9,
irr\((G_a(k)) = 4k\), and \(irr_t(G_a(k)) = 8k^2\). It is interesting to note that \(irr(G_a(k))/n = irr(G_b(k))/n = 4/3\), and \(irr_t(G_a(k))/n^2 = irr_t(G_b(k))/n^2 = 4/9\), for any \(k \geq 3\).

**Example 4.** Another infinite sequence of bidegreed graph pairs denoted by \(H_a(k)\) and \(H_b(k)\) is shown in Fig. 11. Each of them has \(n = 4k\) vertices, where \(k \geq 2\). Graphs \(H_a(k)\) and \(H_b(k)\) are edge-equivalent, because the identities

\[
\begin{align*}
\text{CS}(H_a(k)) = & \text{CS}(H_b(k)) = (\sqrt{17} - 4)/2, \\
\text{Var}(H_a(k)) = & \text{Var}(H_b(k)) = 1/4, \\
\text{irr}(H_a(k))/n = & \text{irr}(H_b(k))/n = 1, \\
\text{irr}_t(H_a(k))/n^2 = & \text{irr}_t(H_b(k))/n^2 = 1/4.
\end{align*}
\]

**Fig. 11.** Bidegreed graph pair \(H_a(k)\) and \(H_b(k)\)

\(m_{2,3} = 4k = n, m_{3,3} = k, \) and \(m = 5k\) hold. It is easy to see that \(H_a(k)\) and \(H_b(k)\) are 2-walk \((1, 4)\) linear graphs. From this it follows that they have identical spectral radius which is equal to \((1 + \sqrt{17})/2\). For graphs \(H_a(k)\) and \(H_b(k)\) the following equalities hold: \(\text{CS}(H_a(k)) = \text{CS}(H_b(k)) = (\sqrt{17} - 4)/2, \)

\[
\begin{align*}
\text{Var}(H_a(k)) = & \text{Var}(H_b(k)) = 1/4, \\
\text{irr}(H_a(k))/n = & \text{irr}(H_b(k))/n = 1, \\
\text{irr}_t(H_a(k))/n^2 = & \text{irr}_t(H_b(k))/n^2 = 1/4.
\end{align*}
\]

**Example 5.** Semi-regular bidegreed graph pairs denoted by \(J_a(k)\) and \(J_b(k)\) are shown in Fig. 12. Both of them are comprised of \(n = 5k\) vertices, where \(k \geq 2\). Graphs \(J_a(k)\) and \(J_b(k)\) are edge-equivalent, since the identity \(m_{2,3} = 6k\) is fulfilled. Moreover, these graphs are 2-walk \((0, 6)\) linear. Consequently, they have identical spectral radius which is equal to \(\sqrt{6}\). For graphs \(J_a(k)\) and \(J_b(k)\) the following equalities hold: \(\text{CS}(J_a(k)) = \text{CS}(J_b(k)) = \sqrt{6} - 12/5, \)

\[
\begin{align*}
\text{Var}(J_a(k)) = & \text{Var}(J_b(k)) = 6/25, \\
\text{irr}(J_a(k))/n = & \text{irr}(J_b(k))/n = 6/5, \\
\text{irr}_t(J_a(k))/n^2 = & \text{irr}_t(J_b(k))/n^2 = 6/25.
\end{align*}
\]

**4.5 Smallest bidegreed graphs with identical irregularity indices**

In this section we present pairs of smallest graphs that have identical two or more irregularity measures. The results were obtained by computer search. For two graphs of same order \(G_1 = (V_1, E_1)\) and \(G_2 = (V_2, E_2)\), we said that \(G_1\) is smaller than \(G_2\) if \(|E_1| < |E_2|\). Consequently, for two pairs of graphs of same order \(D_1 = (G_1, G_2)\) and \(D_2 = (G_3, G_4)\), we said that \(D_1\) is smaller than \(D_2\) if \(|E_1| + |E_2| < |E_3| + |E_4|\).
First, in Fig. 13(a) the smallest pair of graphs, that have identical all four irregularity indices $CS$, $Var$, $irr$ and $irr_t$, is presented. The graphs $G_1$ and $G_2$ are of order 6 and size 7. Their $CS$, $Var$, $irr$ and $irr_t$ indices are 0.080880, 0.266667, 4, and 8, respectively. They also have same spectral radius which is 2.414214. We note that the pair $(G_1, G_2)$ is at same time the smallest pair of graphs with equal $CS$ index.
In Fig. 13(b) the smallest pair of graphs, that have identical Var, irr and irr_{t} indices is presented. This pair is also the smallest pair with the property that both graphs have equal Var and irr indices. The graphs G_3 and G_4 are of order 5 and sizes 6 and 9, respectively. Their Var, irr and irr_{t} indices are 0.300000, 6, and 6, respectively.

The pair (P_5, G_3), depicted in Fig. 13(c), is the smallest pair with the property that both graphs have equal Var and irr_{t} indices. At same time, it is the smallest pair with both graphs having equal Var index. Also, it is the smallest pair with both graphs having equal irr_{t} index. Their Var and irr_{t} indices are 0.300000 and 6, respectively.

The pair (S_5, G_3), depicted in Fig. 13(d), is the smallest pair with the property that both graphs have equal irr and irr_{t} indices. It holds that irr(S_5) = irr(G_3) = 12 and irr_{t}(S_5) = irr_{t}(G_3) = 12. At same time, together with the pair (P_5, G_5), it is the smallest pair with both graphs having equal irr index.

5 Final remarks and open problems

In this paper we focused our investigation to the study of the relations between the irregularity indices of bidegreed connected graphs. Comparing the irregularity indices of various graphs, in the majority of cases it was supposed that the number of vertices or the corresponding degree sets are identical (see Figures 3, 4, 6, 8, 9, 10, 11, 12, 13(a)). It would be interesting to consider graphs of same order which have different degree sets, but their corresponding irregularity indices are identical (as few examples in Fig. 13(b),(c),(d)).

Another interesting problem is to estimate the maximum possible difference of vertex and edge numbers of graphs having identical irregularity indices (assuming that such positive finite integer exists.) Both cases, when graphs are of same or different order, are of interest. In Fig. 14, bidegreed graphs B(6, 5) and B(3, 2) represent an example concerning this problem. We would like to note that, the bidegreed polyhedral graph B(6, 5) is the dual of the
graph of the smallest $C_{24}$ fullerene which is composed of 12 pentagonal and 2 hexagonal faces, and graph $B(3, 2)$ is a semiregular graph. It is worth noting that graph $B(6, 5)$ has 14 vertices and 36 edges, while graph $B(3, 2)$ has 10 vertices and 12 edges. It is surprising that there is a large difference between the corresponding edge-numbers of the two graphs, $(36 - 12 = 24)$.

References