

On the Implementation of Fixed Point Iteration-based Adaptive Receding Horizon Control for Multiple Degree of Freedom, Higher Order Dynamical Systems

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Abstract: Based on the observation that it is very difficult to combine the mathematical frameworks of the prevailing Lyapunov function-based adaptive controllers and the traditional, optimal control-based “Model Predictive Controllers (MPC)”, a novel adaptive solution was introduced to improve the operation of the “Receding Horizon Controllers (RHC)”. Because at the local optima the gradient of the auxiliary function is zero, Lagrange’s original “Reduced Gradient Method (RGM)” was replaced by a “Fixed Point Iteration (FPI)”-based algorithm that directly drove this gradient toward zero. According to “Banach’s Fixed Point Theorem” the convergence of the method was guaranteed by a contractive function that generated the iterative sequence. The greatest modeling burden in this approach was the need for the calculation of the Jacobian of the problem, i.e. the gradient of the gradient of the auxiliary function. In the first simulations only a single “Degree of Freedom (DoF)” 2nd order nonlinear system, a van der Pol oscillator was investigated. The attempts that were made to evade the calculation of the Jacobian were finished with the conclusion that at least a rough numerical approximation of this Jacobian generally must be utilized. Though the MPC approach allows the use a great variety of cost functions and dynamical models, mathematically well established results are available only for quadratic cost functions and “Linear Time-invariant (LTI)” models. For other cost functions and models careful numerical analysis is needed. In this paper the use of non-quadratic cost functions is numerically investigated in the FPI-based adaptive RHC control of 2 DoF 2nd order nonlinear system that consists of two, nonlinearly coupled van der Pol oscillators, is considered. To guarantee lucid calculations simple functions are introduced that map the active parts of the horizon under consideration to the elements of the gradient of the auxiliary function that are calculated analytically. For the calculation of the Jacobian only a rough numerical estimation is applied. The simulation results reveal certain limitations of the suggested method.

Keywords: Optimal Control; Model Predictive Control; Receding Horizon Control; Adaptive Control; Fixed Point Iteration-based Adaptive Control; Banach Space; van der Pol Oscillator

1 Introduction

The “*Optimal Controllers (OC)*” mathematically can be formulated in strict analogy with finding the local extremum of the action functional in Classical Mechanics. The implementation of the variational calculus leads to the Hamilton – Jacobi – Bellman equations, that numerically can be carried out by “*Dynamic Programming (DP)*” introduced by Bellman in the fifties of the past century [1, 2]. Though the computational power of the computers has been tremendously increased by our days, for control applications this approach means a great computational burden, so alternative methods are in use, too. The most popular one is the heuristic RHC introduced in the late seventies of the past century [3] in which the time domain is approximated by a discrete grid, the optimization process is limited to consecutive finite fragments of this grid that are referred to as “horizons”, and it is carried out by Lagrange’s RGM method he invented for the use in Classical Mechanics about 1811 [4]. This approach is often referred to as “*Nonlinear Programming (NP)*”. It allows the use of various cost functions and dynamic models e.g. in chemistry [5, 6], and for its realization for “limited size problems” the professional “Solver” package of Microsoft’s Excel is generally available. While its use in economic calculations is very popular (e.g. [7, 8, 9]), in optimal control problems it is less frequent (e.g. [10] also related to economic and educational issues). Based on these antecedents in [11] the dynamic control of a 1st order single variable hypothetical system, and in [12] its possible use in the model-based treatment of the illness type 1 diabetes mellitus was investigated.

It was observed in the past century [13, 14] that in the case of LTI dynamic models and quadratic cost functions considerable mathematical simplifications are generally available. In the so introduced “*Linear Quadratic Regulator (LQR)*” instead of the original RGM either the differential or the algebraic form of the Riccati equation has to be solved. (This special form can be achieved by seeking the solution of the problem in a special product form so “decoupling” two variables in the first step.) In 1724 Riccati realized that the solution of a special scalar quadratic differential equation can be obtained by solving a linear one [15]. In the case of matrix equations similar possibilities are available by the use of the method of “Schur complements” introduced by Haynsworth in 1968 [16]. The solution of these linear matrix equations can be tackled by the method of “*Linear Matrix Inequalities (LMI)*” for the realization of which efficient software packages are available [17]. For more complicated cases “state-dependent Riccati equations” are in use (e.g. [18]). The LQR method is widely used in nonlinear dynamic systems in which the available nonlinear models are often linearized for the sake of the use of this special formalism. Furthermore, the quadratic structure of the cost functions also means significant limitations.

Since it is a special variant of the “*Model Predictive Control (MPC)*”, the applicability of the RHC also depends on the reliability of the available system models.

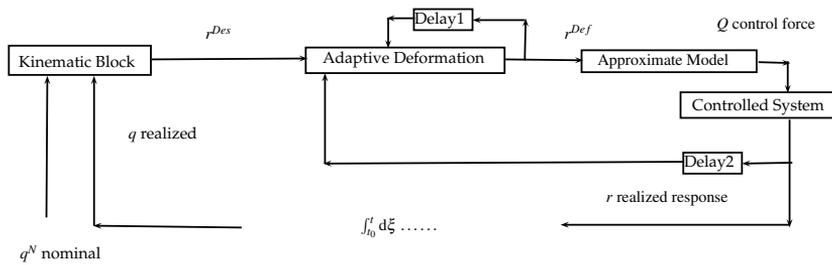


Figure 1

Schematic structure of the “Fixed Point Transformation-based Adaptive Controller” after [19] (the adaptive deformation can be realized by the use of various fixed point transformations, and the system’s response r can be an arbitrary order time-derivative of the generalized coordinates of the controlled system)

The general difficulties of this approach consist in the lack of reliable system models, and limited possibilities for state observation. Illustrative examples related to various subject areas as robotics (e.g. [20, 21]), turbojet engine modeling (i.e. [22, 23, 24, 25]), life sciences as modeling the glucose-insulin system (e.g. [26, 27, 28, 29]), anaesthesia control (e.g. [30, 31, 32, 33, 34, 35]) can be mentioned. To compensate the effect of modeling imprecisions a possibility is the application of structurally precisely known analytical models with unknown parameters that can be tuned via observations by “Adaptive Controllers (AC)”. All the prevailing approaches are based on Lyapunov’s dissertation on the stability of motion he defended in 1892 [36, 37] and appeared in the adaptive control in the nineties of the past century (e.g. [38, 39]). It does not seem to be easy to combine the mathematical framework of the Lyapunov function-based adaptive controllers with that of the cost function-based optimal controllers.

An alternative approach to adaptive control design was suggested in 2009 in [40] outlined in Fig. 1 where the controlled system’s response $r(t) \equiv q^{(n)}(t)$, $n \in \mathbb{N}$, i.e. it is an order n time-derivative of its generalized coordinate q , that is the state variable consists of the components $\{q, \dot{q}, \ddot{q}, \dots, q^{(n-1)}\}$. On the basis of purely kinetic/kinematic considerations a “Desired Response” $r^{Des}(t) \equiv q^{(n)Des}(t)$ can be calculated that is able to drive the tracking error, i.e. the difference between the “Nominal Trajectory” $q^N(t)$ and the realized trajectory $q(t)$ to 0 as $t \rightarrow \infty$ if it is precisely realized. Such a strategy can be formulated e.g. by the use of the “Integrated Tracking Error” with a positive constant Λ as in (1), in which t_0 denotes the starting time of the control:

$$e_{Int}(t) \stackrel{def}{=} \int_{t_0}^t [q^N(\xi) - q(\xi)] d\xi, \quad (1a)$$

$$\left(\frac{d}{dt} + \Lambda\right)^{n+1} e_{Int}(t) \equiv 0 \Rightarrow \quad (1b)$$

$$q^{(n)Des}(t) = q^{N(n)}(t) + \sum_{\ell=1}^{n+1} \left(\frac{(n+1)!}{\ell!(n+1-\ell)!}\right) \Lambda^\ell \left(\frac{d}{dt}\right)^{n+1-\ell} e_{Int}(t). \quad (1c)$$

In the special case of $n = 2$ (1) corresponds to a PID-type error-feedback with the

proportional coefficient $K_P = 3\Lambda^2$, integral coefficient $K_I = \Lambda^3$, and derivative coefficient $K_D = 3\Lambda$. By feeding back the integrated error, if it is precisely realized, (1) makes the tracking error asymptotically converge to 0. However, if it is only *approximately realized*, the integrated tracking error can diverge. In this case the generalization of the PD-type feedback is more successful as defined in (2) as

$$e(t) \stackrel{def}{=} q^N(t) - q(t) , \quad (2a)$$

$$\left(\frac{d}{dt} + \Lambda \right)^n e(t) \equiv 0 \Rightarrow \quad (2b)$$

$$q^{(n)Des}(t) = q^{N^{(n)}}(t) + \sum_{\ell=1}^n \left(\frac{n!}{\ell!(n-\ell)!} \right) \Lambda^\ell \left(\frac{d}{dt} \right)^{n-\ell} e(t) . \quad (2c)$$

According to [41, 42] this feedback strategy was successfully applied for the control of ships in the first half of the past century. In the “*Kinematic Block*” of the FPI-based adaptive controller described in Fig. 1 a combined application of (1) and (2) can be successful: at the beginning, when the tracking error is still big, the controller can use (2), and later it can turn to (1) when the use of the integrated term causes more precise tracking instead of divergence.

Returning back to the scheme in Fig. 1, to compensate the effects of the modeling errors, before using the available “*Approximate Model*” for the calculation of the force or the other appropriate control signal that is necessary for producing $q^{(n)Des}(t)$, its “deformed version” $q^{(n)Def}(t)$ is calculated in the block “*Adaptive Deformation*”. In the case of digital controllers this calculation happens in discrete control cycles, and the two “*Delay*” blocks correspond to a single control cycle: in the given cycle the system “learns” from the deformed signal applied, and the system’s response obtained in the previous cycle. It is important to see that in the case of an order n physical system $q^{(n)}$ can have very fast variation because it is directly set by the control force $Q(t)$, but the “desired” value $q^{(n)Des}(t)$ varies only slowly for moderate $q^{N^{(n)}}(t)$ because it is constructed of the integrated tracking error, and the order $\{0, 1, \dots, n-1\}$ time-derivatives of the tracking error. In the block “*Adaptive Deformation*” a function of the form $q_{i+1}^{(n)Def} = G(q_i^{(n)Def}, q_i^{(n)}, q_{i+1}^{(n)Des})$ is applied ($q_i^{(n)Def}$ is fed back at the input on the top) so that for constant $q^{(n)Des}$ it could generate a sequence of deformed signals as $\{q_1^{(n)Def}, q_2^{(n)Def} = \Phi(q_1^{(n)Def}), \dots, q_{i+1}^{(n)Def} = \Phi(q_i^{(n)Def}), \dots\}$ since in a given state the controlled system’s response depends on $q^{(n)Def}$. With other words, during a digital control step only one step of the iteration can be realized. The possible convergence of the iteration was investigated on the basis of Banach’s “*Fixed Point Theorem*” [43] according to which in a linear, normed, complete metric space a contractive map generates a sequence that converges to the unique fixed point of this map. For this purpose the contractivity of the function $\Phi(q^{(n)Def})$ must be guaranteed. In [40, 44, 45] various solutions were suggested for the function G .

It must be emphasized that this approach may provide satisfactory results if the generated sequence has fast convergence and the slowly varying “desired” value $q^{(n)Des}$ “pulls with itself” the limit point of the convergence. The applicability of this approach in each particular case needs preliminary investigations via simulation before trying any real application. For instance, in the adaptive control of a brushless electric DC motor the method was found to be successful via simulations and experimental investigations [46]. In the case of an inverted pendulum-type electric cart via simulations it was found that this approach may be successful for controlling the tilting angle, but in the control of its linear displacement it could cause not tolerable, tremendous spinning up [47].

Returning to the question of adaptive MPC or adaptive RHC in [11] a simple proposition was done. While in the original version of the RHC the problem of the modeling errors is tackled by the application of the available approximate system model for the calculation of the control signals, and normally only a small part of the calculated horizon is actually utilized, furthermore, the control is frequently redesigned by the introduction of novel horizons in the beginning of which the actually measured system variables are used as initial conditions, in [11] the following idea was suggested: instead using the control forces generated by the RHC controller, the controller can adaptively track the “optimized trajectory”. In this approach the mathematical frameworks of the optimal and the adaptive controllers can be integrated without any technical difficulty.

A further step towards the FPI-based adaptive RHC was based on the observation that Lagrange’s RGM may be replaced by a fixed point iteration in the optimization phase of the calculations [48, 49]. The idea was generated by considering analogies with the problem of solving the inverse kinematic task for redundant robot arms.

The prevalent literature in robotics applies an augmented Jacobian for obstacle avoidance (e.g. [50, 51]) and uses the Moore-Penrose pseudoinverse [52, 53] for the disambiguation of the otherwise ambiguous solution of this task. To utilize the advantages of the ambiguity of the solution to this special choice some elements of the null space of the Jacobian are added later (e.g. [54, 55, 56]) that makes the problem of the continuity of the so obtained solution arise. Furthermore, this pseudoinverse is calculated by the utilization of the traditional inverse of a quadratic matrix that is not invertible in the singular configurations, and is ill conditioned in their vicinity. To treat this problem the singular matrix is completed by an addendum that makes it invertible, that corresponds to a “deformation” of the original task: “*Damped Least Squares (DLS)*” [57, 58]. Desperate attempts were invented in the early nineties of the past century to tackle the problem of singularities as considering 2nd order differential inverse kinematics [59] or the complex extension of the real generalizes coordinates in [60]. In [61] D. Drexler suggested a witty idea of “action point transformation” for the regularization of the Jacobian, and in [62] an implicit 2nd order integration was suggested.

Realizing the fact that the Moore-Penrose pseudoinverse can simply be generalized by the use of different weights for the quadratic cost function contributions that play similar role as the “joint activation factors” when certain elements of the null space

of the Jacobian are mixed into the solution after computing the pseudoinverse, a matrix inversion-free, FPI-based solution of the inverse kinematic task was proposed by Csanádi *et al.* in [63]. The forward kinematic task is expressed by the function $f: \mathbb{R}^n \mapsto \mathbb{R}^m$ in which $n, m \in \mathbb{N}$, and $n > m$. The “initial position” $x_{ini} = f(q_{ini})$ is known, and a function of the scalar variable $t \in \mathbb{R}$, $q(t)$ has to be found for which $x(t) = f(q(t))$, $x(t_{ini}) = x_{ini}$, and $q(t_{ini}) = q_{ini}$. The *differential form* of this task is

$$\dot{x}(t) = \frac{\partial x(q(t))}{\partial q} \dot{q}(t) \equiv J(q(t)) \dot{q}(t) . \quad (3)$$

Instead of computing the generalized inverse of $J(q)$, Csanádi introduced a finite time grid with the resolution Δt , considered a given configuration at time i as $x(t_i), q(t_i)$, computed the next position as $x(t_i + \Delta t)$, and commenced an “internal iteration” between the neighboring grid points as

$$\begin{aligned} q(t_i + \Delta t, s + 1) &= G(q(t_i + \Delta t, s), f(q(t_i + \Delta t, s)), x(t_i + \Delta t)) , \\ q(t_i + \Delta t, 1) &= q(t_i) \end{aligned} \quad (4)$$

that is inserted the task of finding the differential solution into the FPI-based adaptive control scheme depicted in Fig. 1. It soon became clear that a common Jacobian generally will not produce convergent sequence, so the task was modified as

$$J^T(q(t))x(t) = J^T(q(t))f(q(t)) , \text{ or on the grid} \quad (5a)$$

$$J^T(q(t_i))x(t_{i+1}) \approx J^T(q(t_i)) \left[f(q(t_i)) + \frac{\partial f}{\partial q} \Big|_{q(t_i)} (q(t) - q(t_i)) \right] , \quad (5b)$$

in which a 1st order Taylor series approximation of function $f(q)$ was taken in $q(t_i)$. This task corresponds to the iteration

$$J^T(q(t_i)) [x(t_{i+1}) - f(q(t_i))] \approx J^T(q(t_i)) J(q(t_i)) (q - q(t_i)) . \quad (6)$$

In a nonsingular configuration the positive definiteness of the matrix $J^T(q(t_i))J(q(t_i))$ guaranteed the convergence of the internal iteration, while in and in the vicinity of the singular configurations it automatically caused the “stagnation” of the appropriate axes until the vicinity of the singularity was left. That is, this approach automatically treated the problem of the singular configurations without the use of any complementary trick or task deformation, and it yielded nice solutions without risking the application of big joint coordinate time-derivatives.

If the same task is tackled on the basis of Lagrange’s RGM method, the optimization under constraints can be formulated in (7)

$$\Phi \stackrel{def}{=} \sum_i P_i \dot{q}_i^2 = \text{extremum under the constraints} \quad (7a)$$

$$\dot{x}_k - \sum_\ell J_{k\ell}(q) \dot{q}_\ell = 0 \quad \forall k , \quad (7b)$$

in which the weights $P_i > 0$ express the “penalty” for rotating the axle i in the solution. As is well known, the *auxiliary function* of the above problem is

$$\Psi(\dot{q}, \lambda) = \sum_i P_i \dot{q}_i^2 + \sum_k \lambda_k \left(\dot{x}_k - \sum_\ell J_{k\ell} \dot{q}_\ell \right), \quad (8)$$

in which the symbols λ_k denote the Lagrange multipliers, and in the local optima $\forall i, j$ it holds that $\frac{\partial \Psi}{\partial \dot{q}_i} = 0$ and $\frac{\partial \Psi}{\partial \lambda_j} = 0$.

In [48, 49] it was recognized that driving the array $\nabla \Psi$ toward 0 is a task very similar to that formulated in (4) and (5). It was also recognized that for the validity of this structure *it is not necessary to have quadratic contributions in the cost function*. The question that $\nabla \Psi = 0$ in both the local minima and maxima of the cost function was evaded by the following argumentations:

1. In the calculation of the Moore-Penrose generalized inverse, as well as in the case of the modified version in (7), immediately matrix equations are provided after computing $\nabla \Psi$. These equations can be solved without minding if the solution belongs to a local minimum or maximum of Φ . However, it is known that the global minimum of Φ is zero that belongs to $\dot{q} = 0$ that generally does not satisfy the constraint equations in (7b). Also, it is clear that Φ does not have local maximum, it can diverge to ∞ for infinitely large \dot{q} components. Therefore the iteration that is commenced from a low $\|\dot{q}\|$ will increase this norm until the reduced gradient of the problem, i.e. $\nabla \Psi$ achieves 0.
2. In Classical Thermodynamics (e.g. [64]) in the state of the thermal equilibrium the entropy of an isolated system must have its maximum under the constraint of fixed internal energy, and all the other constraints determined by the internal thermodynamic walls that partly isolate the subsystems of the big isolated system under consideration (the “*Entropy Maximum Principle*”). The entropy is monotonic increasing, unbounded function of the internal energy. Again, in finding the thermal equilibrium the zero value of the appropriate $\nabla \Psi$ is found, in which the Lagrange multipliers have definite physical interpretation.
3. Again, in Classical Thermodynamics, the “*Energy Minimum Principle*” states that in the state of thermal equilibrium the internal energy of the big isolated system takes its minimum value under the main constraint of the fixed entropy and all the other constraints determined by the internal thermodynamic walls that partly isolate the subsystems of the big isolated system under consideration. Again, the fixed entropy does not allow to reach the absolute minimum of the energy that in principle is allowed by Classical Physics. The internal energy does not have upper bound. In the solution the Lagrange multipliers have definite physical interpretation, and if a certain subsystem behaves as a reservoir, i.e. it keeps the values of certain Lagrange multipliers constant, in the formalism automatically appear the appropriate Legendre transform of the internal energy that offers the introduction of the thermodynamic potentials as the enthalpy, the Helmholtz or the Gibbs potentials. Again, the question of local minima or maxima generally is not pondered.

In the field of Optimal Control the tracking errors at the internal points of the horizon, and at the last point of the horizon, the control force components, and the constraint equations that express the dynamic properties of the controlled system together with the Lagrange multipliers provide the components of the gradient of the appropriate auxiliary function. The norm of the tracking error does not have theoretical upper bound, but it has the value 0 as the theoretical lower limit. In principle the control force components can have infinite values, so the situation is similar to that of the above considered general examples.

In [48, 49] the components of $\nabla\Psi$ and that of the Jacobian, i.e. $\nabla\nabla\Psi$ were analytically computed for single DoF systems that was found to be a great programming burden. In [65] an attempt was made to evade the calculation of the Jacobian but it lead to the conclusion that at least some numerical estimation of this matrix practically is needed to achieve convergence.

In the present paper the method is further investigated by numerical simulations for a strongly nonlinear, 2nd order, 2 DoF dynamical system and non-quadratic cost functions. Only the components of $\nabla\Psi$ are analytically calculated, the Jacobian is obtained by rough numerical approximations, due to the complexity of the programming task. In the sequel the dynamic model of the controlled system, the generation of the nominal trajectory, the cost functions, and the FPI-based adaptive RHC are investigated via numerical simulations.

2 The Dynamic Model of the Two Coupled van der Pol Oscillators

The model of the van der Pol oscillator [66] originally was developed to describe the nonlinear oscillations of an externally excited triode in 1927. For the sake of simplicity in this paper it is interpreted as a “mechanical system” with the equations of motion given in (9)

$$\ddot{q}_1 = (-k_1(q_1 - L_{10}) - b_1((q_1 - L_{10})^2 - a_1^2)\dot{q}_1 + k_{12}g(q_1, q_2) + F_1)/m_1 , \quad (9a)$$

$$\ddot{q}_2 = (-k_2(q_2 - L_{20}) - b_2((q_2 - L_{20})^2 - a_2^2)\dot{q}_2 - k_{12}g(q_1, q_2) + F_2)/m_2 , \quad (9b)$$

$$g(q_1, q_2) = |q_2 - q_1 - L_{12}|^\sigma \text{sign}(q_2 - q_1 - L_{12}) , \quad (9c)$$

with the parameter values given in Table 1. The generalized coordinates are $q_1 [m]$ and $q_2 [m]$, and $F_1 [N]$ and $F_2 [N]$ denote the control forces. The different signs of the coupling forces in (9a) and (9b) expresses the fact that they are each other’s reaction forces. The “*excitation/damping separator*” parameters separate from each other the excited and the damped regions. If no interaction occurs between the oscillators (i.e. $k_{12} = 0$) the $q_1 = L_{10}$, $q_2 = L_{20}$, $\dot{q}_1 = 0$, and $\dot{q}_2 = 0$ state corresponds to an unstable equilibrium point of these oscillators. Even an infinitesimally small perturbation that kicks the system out of this equilibrium point causes nonlinear oscillations that are bounded because for too big amplitudes the excitation terms turn into damping factors. The motion of the free oscillators is depicted in Figs. 2

and 3. In the control simulation the trajectory depicted in Fig. 2 was in use as the “Nominal Trajectory”.

Table 1
The applied exact and approximate model parameters

Parameter	Exact	Approximate
m_1 [kg] mass	1.0	$0.5 \cdot m_1$
m_2 [kg] mass	2.0	$1.2 \cdot m_2$
k_1 [$N \cdot m^{-1}$] spring constant	10.0	$1.2 \cdot k_1$
k_2 [$N \cdot m^{-1}$] spring constant	15.0	$0.8 \cdot k_2$
L_{10} [m] zero force position	-5.0	$1.3 \cdot L_{10}$
L_{20} [m] zero force position	3.0	$0.9 \cdot L_{20}$
b_1 [$N \cdot s \cdot m^{-1}$] viscous damping/excitation coeff.	1.2	$1.2 \cdot b_1$
b_2 [$N \cdot s \cdot m^{-1}$] viscous damping/excitation coeff.	1.3	$1.2 \cdot b_2$
a_1 [m] excitation/damping separator	$\sqrt{0.4}$	$0.5 \cdot a_1$
a_2 [m] excitation/damping separator	$\sqrt{0.5}$	$1.2 \cdot a_2$
σ [nondim.] nonlinearity exponent	1.2	$1.2 \cdot \sigma$
k_{12} [$N \cdot m^{-\sigma}$] nonlinear stiffness	16.0	$1.1 \cdot k_{12}$

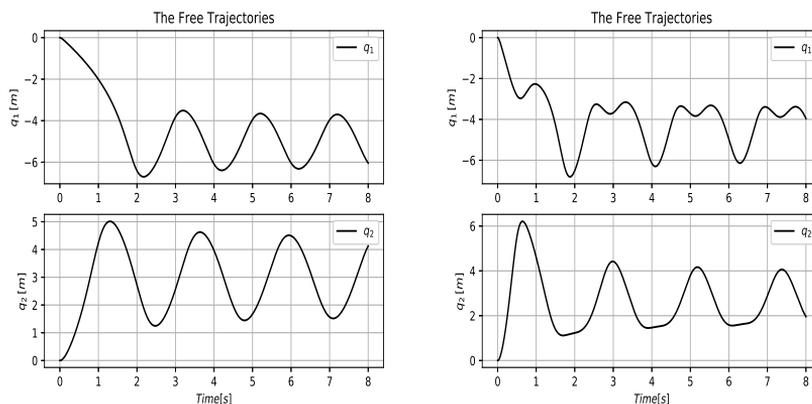


Figure 2

The trajectories of the free motion ($F_1 \equiv F_2 \equiv 0$) of the decoupled oscillators (i.e. the $k_{12} = 0$ case) (LHS) and that of the coupled one obtained by Euler integration of (9) using the “Exact” parameters and a fixed time-step $\delta t = 10^{-3}$ [s]

3 The Adaptive RHC Controller

In the numerical simulations in a horizon 6 grid points were applied of which the 1st and the 2nd one determined the *initial conditions* of the 2nd order dynamical system. The constraints within the horizon were formulated by the approximation as $(q_H(i+2) - 2q_H(i+1) + q_H(i))/\delta t^2 - \ddot{q}_H(i) = 0$, i.e. the optimization task

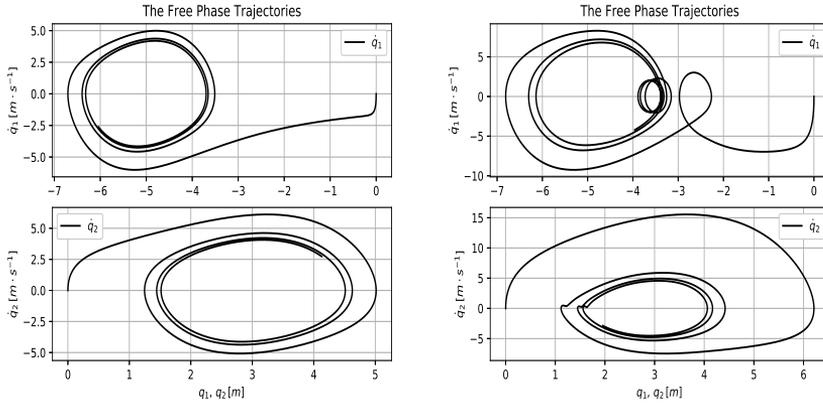


Figure 3

The phase trajectories of the free motion ($F_1 \equiv F_2 \equiv 0$) of the decoupled (LHS) and the coupled (RHS) oscillators (i.e. when k_{12} takes the “Exact” value in Table 1) obtained by Euler integration of (9) using the “Exact” parameters and a fixed time-step $\delta t = 10^{-3}$ [s]

had 24 variables within a horizon as from x_1 to x_4 as $q_{H1}(3), \dots, q_{H1}(6)$, from x_5 to x_8 as $q_{H2}(3), \dots, q_{H2}(6)$, from x_9 to x_{12} as $F_{H1}(1), \dots, F_{H1}(4)$, from x_{13} to x_{16} as $F_{H2}(1), \dots, F_{H2}(4)$, from x_{17} to x_{20} as $\lambda_{H1}(1), \dots, \lambda_{H1}(4)$, and finally from x_{21} to x_{24} as $\lambda_{H2}(1), \dots, \lambda_{H2}(4)$. For feeding the input of the model in (9) the numerical differentiator functions as $der1(q_H(i+1), q_H(i)) = (q_H(i+1) - q_H(i)) / \delta t$ were introduced in a Julia [67] code. The cost functions were defined for the horizon points $i = 3, 4, 5$ as $C(q^N, q^O) = \log \left(1 + M \left| \frac{q^N - q^O}{A_C} \right|^{\alpha_C} \right)$, for the terminal point $i = 6$ as $T(q^N, q^O) = \log \left(1 + M \left| \frac{q^N - q^O}{A_T} \right|^{\alpha_T} \right)$ (the superscript N denotes the nominal trajectory, while O belongs to the optimized trajectory), and for the control force components for $i = 1, 2, 3, 4$ the cost function is $P(F^O) = B_u \log \left(1 + M \left| \frac{F^O}{A_F} \right|^{\alpha_F} \right)$ in which the parameters have lucid geometric interpretations as A_C is responsible for the “strictness” of tracking: if $\alpha_C > 1$ the penalty is small for $|q^N - q^O| < A_C$, and strongly increases for $|q^N - q^O| > A_C$. The parameter α_C determines the “sharpness” of the increase of the penalty function. Similar interpretation can be given to the parameters that limit the applicable control force components. The role of parameter $M > 0$ is “softening” the penalty function for the too big error or force components within the function $\log(\cdot)$.

In the simulations different adaptive deformation functions were applied in the appropriate block of Fig. 1. Both versions were introduced by Dineva in [44, 45] as

$$F : \mathbb{R} \mapsto \mathbb{R} \text{ defined as } F(x) = \text{atanh}(\tanh(x + D_{ad})/2) ,$$

$$F(x_\star) = x_\star , \quad h_n \stackrel{\text{def}}{=} r_n - r_{n+1}^{\text{Des}} , \quad (10)$$

$$r_{n+1}^{\text{Def}} = (F(x_\star + A_c \|h_n\|) - x_\star) \frac{h_n}{\|h_n\| + \varepsilon} + r_n^{\text{Def}}$$

that was used for the optimization of the horizon, and a similar one in which in the role of the scalar function $F(x) = \frac{x}{2} + D_{ad}$ was applied for the adaptive tracking. In both functions $D_{ad} = 0.3$ was set. For the optimization, a very cautious value $A_c = -10^{-3}$, and for the adaptive trajectory tracking $A_c = -1.0$ was chosen. The role of the small parameter $\varepsilon = 10^{-10}$ was the evasion of division by zero in the calculation of the unit vector in (10). In the sequel simulation results are presented. The Jacobian was approximated by separately increasing the components of x with $\delta x = 10^{-10}$ and the partial derivatives were estimated accordingly. The last two points of a given horizon were used as the first two initial points of the next horizon. In the optimization process the Lagrange multipliers and the control forces started from the 0 values, and the optimized trajectories were initiated with the nominal ones.

4 Simulation Results

In the 1st step it is investigated that how the cost functions can cooperate with the finite element approximation of the constraints, i.e. the dynamic model of the controlled system. The tracking parameters were $A_C = A_T = 10^{-2}$, $\alpha_C = \alpha_T = 6.0$, the penalization of the control forces was switched off by $B_u = 0$, the ‘‘Approximate’’ model parameters were identical with the ‘‘Exact’’ ones and in both of them $k_{12} = 0$ was set. Only one step was made in the internal iteration, and the adaptive dynamic tracking was switched off. In the cost functions $M = 10^{-1}$ taming parameter was used. In this case it was expected that the tracking of the nominal trajectory could be possible. The tracking results in Fig. 4 reveal that an accumulated delay was produced in the tracking of the nominal trajectory that partly might be caused by the finite element approximation of the 2nd time-derivatives in the constraints, and partly by the taming parameter in the cost function. Following the removal of ‘‘cost function taming’’ (i.e. directly using the form $C(q^N, q^O) = \left| \frac{q^N - q^O}{A_C} \right|^{\alpha_C}$) practically the same result were obtained. *This fact reveals a significant weak point of the RHC-based approach in the case of higher order dynamical systems, because it roots in the finite element approximation of the higher order time-derivatives.*

Observing that the ‘‘optimized’’ trajectories remain in the vicinity of the ‘‘realized ones’’ since the optimization is initiated from the actual initial values, in the adaptive tracking control instead of the ‘‘optimized’’ ones the trajectories estimated by the dynamic model of the optimization (denoted in the figures as $q^{RealEst}$) can be adaptively tracked. These signals are far smoother than the ‘‘optimized’’ trajectory.

The effect of adaptivity in the dynamic trajectory tracking is revealed by Fig. 5. In this case the ‘‘Approximate Model Values’’ were restored, as well as the exact and approximate values of k_{12} . (It has to be remarked that in the beginning a PD-type tracking defined in (2) was applied that later was switched to a PID-type tracking defined in (1) with $\Lambda = 6.0 [s^{-1}]$, because in the initial part the PID-type strategy always caused divergence.) The significance of the coupling force and the modeling errors as well as that of the adaptive dynamic tracking are well illustrated by these figures. Figure 6 shows that the iteration reduced the norm of $\|\nabla\Psi\|$, and quite con-

siderable difference can be revealed between the “optimized” and the “adaptively set” control forces.

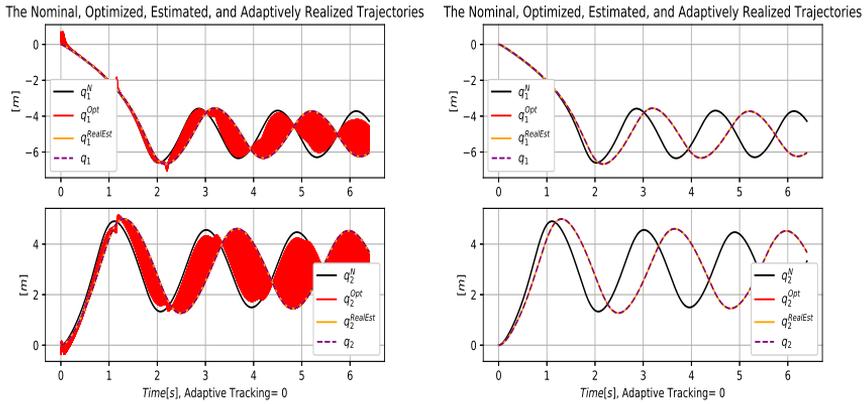


Figure 4 The test trajectory tracking for the decoupled oscillators with (LHS) and without (RHS) showing the optimized trajectory

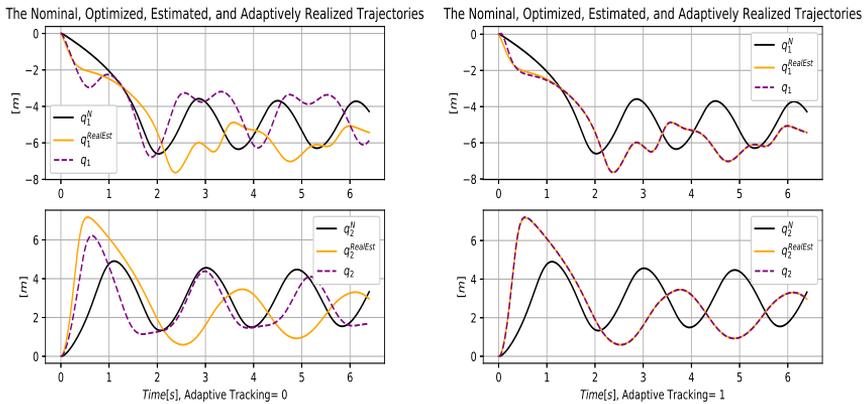


Figure 5 Tracking of the trajectories estimated by the dynamic model of the optimization without (LHS) and with (RHS) adaptivity

In Fig. 7 simulation results are given for $B_u = 10^4$, $A_F = 1.0$, $\alpha_F = 6.0$, and 100 steps in the internal iteration (in which after 10 steps the Jacobian was re-estimated). It can well be seen that the adaptive RHC controller’s parameters behave according to the qualitative expectations. Figure 8 testifies that the adaptive deformation of the 2nd time-derivatives worked correctly, and that the eigenvalues of the estimated $J^T J$ matrix vary over a huge range that results in slow convergence for the smaller eigenvalues.

Finally the adaptive tracking with the modified parameters $A_C = 10^{-3}$, $\alpha_C = 8.0$,

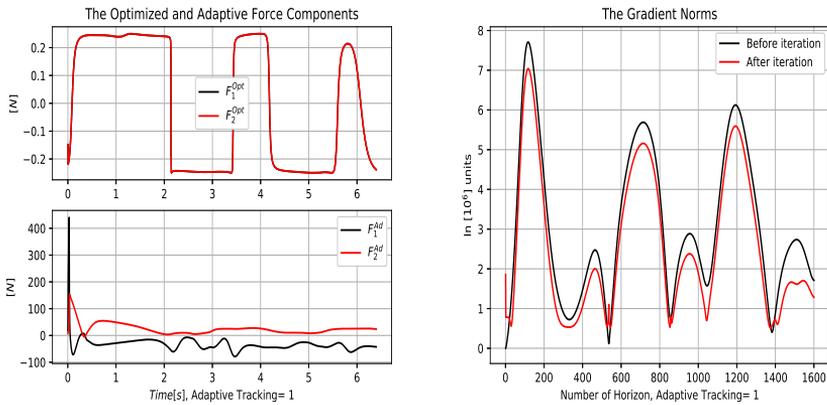


Figure 6
The “optimized” and the adaptively set control forces (LHS), and $\|\nabla\Psi\|$ before and after the optimizing iteration

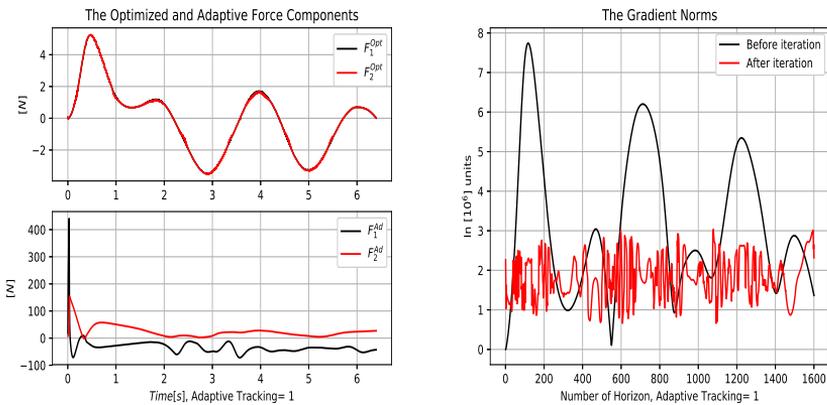


Figure 7
The “optimized” and the adaptively set control forces (LHS), and $\|\nabla\Psi\|$ before and after 100 steps of the optimizing iteration

$A_T = 10^{-3}$, $\alpha_T = 8.0$, $A_F = 10^2$, $\alpha_F = 6.0$, $B_u = 10^4$ and 21 internal steps in the iteration were investigated. Figure 9 reveals that $\|\nabla\Psi\|$ can be reduced essentially during 21 steps, and good adaptive tracking can be achieved. Figure 10 provides information on the Lagrange multipliers, the “optimized” and the adaptive force components.

Conclusions

In this paper the operation of a novel, fixed point iteration-based adaptive RHC was investigated for a 2 DoF 2nd order dynamical system consisting of two, nonlinearly coupled van der Pol oscillators. The method uses essentially the same iteration (with

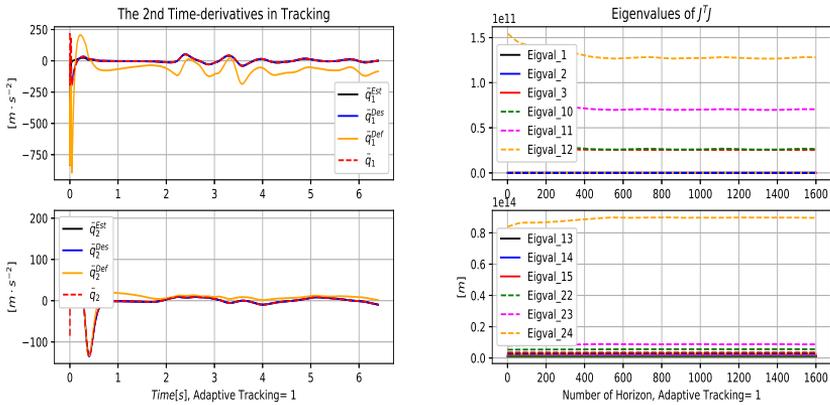


Figure 8 The 2nd time-derivatives of the adaptive tracking (LHS), and the 24 eigenvalues of the estimated $J^T J$ matrix

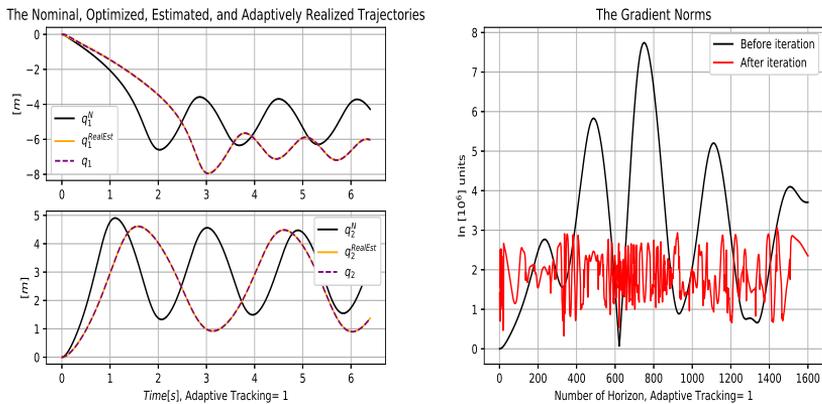


Figure 9 Adaptive trajectory tracking (LHS), and $\|\nabla \Psi\|$ before and after 21 steps of the optimizing iteration

different parameters) for the calculation of the “optimized” trajectory and for the adaptive tracking of the trajectory that was calculated by the approximate dynamic model also used in the optimization.

In the simulations only the gradient of the auxiliary function of the problem was analytically calculated by the use of simple internal functions that map the active (i.e. variable) quantities of the horizon to the elements of this gradient and vice versa. To evade programming complications the Jacobian of the problem was only “roughly”, numerically estimated.

It was found via simulations that the main limitation of the present approach roots in the finite element approximation of the 2nd time-derivatives of the optimized tra-

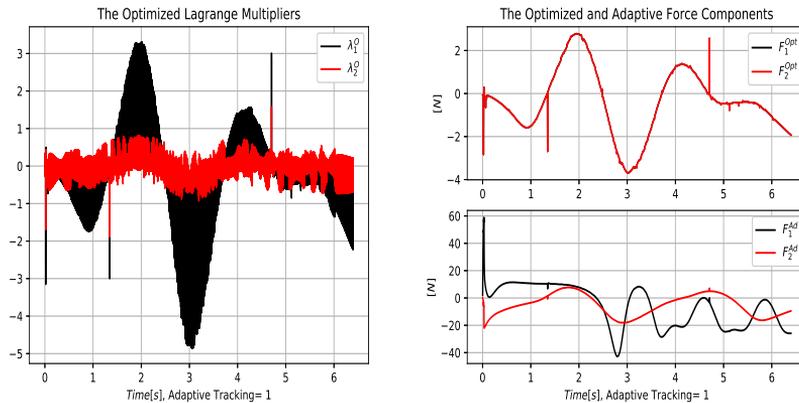


Figure 10

The Lagrange multipliers of the adaptive trajectory tracking (LHS), and the control forces (RHS) using 21 steps in the optimizing iteration

jectory used in the constraint terms of the optimization. With the exception of this problem the simulations supported the qualitative expectations based on the mathematical structure of the suggested method.

The present simulations were restricted to a relatively “short” horizon consisting of 6 grid points of which only 4 points were “active”. It can be assumed that by increasing the horizon length the optimization gets more possibility for improving the “delay” caused by the applied numerical tackling of the constraints. The method deserves more numerical investigations.

Acknowledgement

This work was supported by the *Doctoral School of Applied Informatics and Applied Mathematics* of Óbuda University.

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