Investigation of Failure Systems

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Abstract: In an earlier paper A. Horváth and A. Prékopa [3] applied the Boole-Bonferroni lower and upper bounds to determine the expected time to failure of systems. The main goal of this paper is to show that the so called hypermultitree bounds developed by J. Bukszár [1] also can be applied for investigation of the expected time to failure systems.

Keywords: Boolean-Bonferroni bounds, hypermultitree bounds, time failor systems

1 Introduction

The reliability systems investigated in this paper belong to the field of serial and parallel interconnected systems. Serial system operates if and only if its all components operate. Parallel system operates if and only if at least one components of it operates.

Let us suppose that the components operate or do not operate independently from each other. Let p_i be the operational probability of component i, then the operational probability of serial and parallel systems can be given by the following formulas:

$$r = p_1 p_2 \cdots p_n \tag{1}$$

$$r = 1 - (1 - p_1)(1 - p_2)\cdots(1 - p_n) \tag{2}$$

In the practice one should investigate components with the random operating times, too. The investigation is simpler if they are independent random variables. Let X_i designate the lifespan of component i and $F_i(t)$ its distribution function:

$$F_i(t) = P(X_i \le t), \quad i = 1, \ldots, n. \tag{3}$$

Then with the notations

$$p_i = 1 - F_i(t), \quad i = 1, \ldots, n, \tag{4}$$

formulas (1) and (2) give the operational probability of the system at time t.

In practice besides determining the probability of reliable working state of the system at a certain time, we want to determine other data as well. One of the most important characteristics of this type is the expected value of elapsed time until failure. In the case of serial systems, the elapsed time till the failure of the system equals:

$$X = \min(X_1, X_2, \ldots, X_n) \tag{5}$$

and in the case of parallel systems it equals:

$$Y = \max(X_1, X_2, \ldots, X_n). \tag{6}$$

As it was pointed out in the paper [3], the expected value of these random variables can be determined in the following way.

If a nonnegative random variable Z has probability distribution function $G(z)$, then it is well known that

$$E(Z) = \int_0^\infty [1 - G(z)]\, dz. \tag{7}$$

Using this formula, the expected value of the elapsed time until failure for serial systems can be calculated as

$$\int_0^\infty [1 - F(t)]\, dt, \tag{8}$$

where

$$\begin{aligned} F(t) &= P(X \le t) = \\ &= P(\min(X_1, \ldots, X_n) \le t) = \\ &= 1 - P(\min(X_1, \ldots, X_n) > t) = \\ &= 1 - P(X_1 > t, \ldots, X_n > t) = \\ &= 1 - P(X_1 > t) \cdots P(X_n > t) = \\ &= 1 - (1 - F_1(t)) \cdots (1 - F_n(t)). \end{aligned} \tag{9}$$

In the case of parallel systems the same value can be calculated as

$$\int_0^\infty [1 - G(t)]\, dt, \tag{10}$$

where

$$
\begin{aligned}
G(t) &= P(Y \le t) = \\
&= P(\max(X_1, \dots, X_n) \le t) = \\
&= P(X_1 \le t, \dots, X_n \le t) = \\
&= P(X_1 \le t) \cdots P(X_n \le t) = \\
&= F_1(t) \cdots F_n(t).
\end{aligned}
\tag{11}
$$

The situation is more complicated when the random variables $X_1, X_2, \dots, X_n$ are stochastically dependent. Paper [3] pointed out that in this case one can use the so called Boole-Bonferroni bounds (see for example book [2] by A. Prékopa) to give good lower and upper bounds on the expected value of the elapsed time until failure of systems. In the next section we shortly define the hypermultitree probability bounds introduced by J. Bukszár [1] and in the last section we will show how can be applied these bounds in this context. We remark, that the hypermultitree probability bounds need less calculations and usually are more accurate than the Boole-Bonferroni bounds, so their application in this context may become extremely useful.

2 The Hypermultitree Probability Bounds

In the paper [1] J. Bukszár introduced the concept of (h, m)-hypermultitrees and based on this concept he developed good lower and upper bounds on the probability of union (resp. intersection) of events. As a possible application he estimated the value of the multivariate normal probability distribution function by his newly introduced probability bounds. The definition of the $\Delta = (V,\ {}_h\varepsilon_2, \dots,\ {}_h\varepsilon_{m+1})$, (h, m)-hypermultitree is given in Definition 3 of paper [1]. In the definition V is the set of vertices and ${}_h\varepsilon_i$'s are sets of hyperedges containing $h + i$ vertices. Definition 4 of paper [1] introduces the concept of the weight of (h, m)-hypermultitrees in the following way. Let $A_1, A_2, \dots, A_n$ be arbitrary events and suppose we can calculate the probability of their intersections up to $h + m + 1$ number of events involved in the intersection. Then the weight of the (h, m)-hypermultitree $\Delta = (V,\ {}_h\varepsilon_2, \dots,\ {}_h\varepsilon_{m+1})$ is given by the formula

$$
\begin{aligned}
w(\Delta) \quad = & \quad \sum_{(i_1,\ldots,i_{h+2})\in {}_h\varepsilon_2} P\left(A_{i_1}\cdots A_{i_{h+2}}\right) \\
- & \quad \sum_{(i_1,\ldots,i_{h+3})\in {}_h\varepsilon_3} P\left(A_{i_1}\cdots A_{i_{h+3}}\right) \\
\vdots & \\
+ & \quad (-1)^{m+1} P\left(A_{i_1}\cdots A_{i_{h+m+1}}\right).
\end{aligned}
\tag{12}
$$

J. Bukszár proved the following inequalities in paper [1] (see Theorem 1):

If $A_1, A_2, \ldots, A_n$ are arbitrary events, and $\Delta = (V,\ {}_h\varepsilon_2, \ldots,\ {}_h\varepsilon_{m+1})$ is an arbitrary (h, m)-hypermultitree then the following inequalities hold:

(i) if h is even, then

$$P(A_1 + \cdots + A_n) \le \sum_{k=1}^{h+1} (-1)^{k-1} S_k - w(\Delta), \tag{13}$$

(ii) if h is odd, then

$$P(A_1 + \cdots + A_n) \ge \sum_{k=1}^{h+1} (-1)^{k-1} S_k + w(\Delta), \tag{14}$$

where

$$S_k = \sum_{1\le i_1 < \cdots < i_k \le n} P\left(A_{i_1}\cdots A_{i_k}\right). \tag{15}$$

From these formulae one can see that the probability bounds given by the (h, m)-hypermultitrees are closer to the exact probability value when their weight is larger. The problem of finding the best possible hypermultitree is NP hard, however J. Bukszár in Section 3 of his paper [1] developed very efficient algorithms for finding hypermultitrees with heavy weight. In the next section we will use only the special cases $h = 0$ and $h = 1$ as it was proposed by J. Bukszár.

3 Application of the Hypermultitree Probability Bounds for Investigation of Failure Systems

In the case of serial systems we can apply the formulae (13) and (14) in a straightforward way for the events $A_i = \{X_i \le t\},\ i = 1, \ldots, n,$ as we have

$$F(t) = P(X \le t) = P(\min(X_1, \ldots, X_n) \le t) = P(A_1 + \cdots + A_n). \tag{16}$$

To do this first we introduce the notations:

$$F_{i_1,\ldots,i_k}(t) = P\left(X_{i_1} \le t,\ldots,X_{i_k} \le t\right), 1 \le i_1 < \cdots < i_k \le n, \tag{17}$$

$$S_k(t) = \sum_{1 \le i_1 < \cdots < i_k \le n} F_{i_1,\ldots,i_k}(t), \; k = 1,\ldots,n. \tag{18}$$

Now for $h = 0$ we get the upper bound:

$$\begin{aligned} F(t) \;\; & \le \;\; S_1(t) \\ & - \sum_{(i_1,i_2)\in_0 \varepsilon_2} F_{i_1,i_2}(t) \\ & + \sum_{(i_1,i_2,i_3)\in_0 \varepsilon_3} F_{i_1,i_2,i_3}(t) \\ & \vdots \\ & + (-1)^m \sum_{(i_1,\ldots,i_{m+1})\in_0 \varepsilon_{m+1}} F_{i_1,\ldots,i_{m+1}}(t), \end{aligned} \tag{19}$$

and for $h = 1$ we get the lower bound:

$$\begin{aligned} F(t) \;\; & \ge \;\; S_1(t) - S_2(t) \\ & + \sum_{(i_1,i_2,i_3)\in_1 \varepsilon_2} F_{i_1,i_2,i_3}(t) \\ & - \\ & \vdots \\ & + (-1)^{m+1} \sum_{(i_1,\ldots,i_{m+2})\in_1 \varepsilon_{m+1}} F_{i_1,\ldots,i_{m+2}}(t). \end{aligned} \tag{20}$$

In the case of parallel systems we have

$$1 - G(t) = P(Y > t) = P(\max(X_1,\ldots,X_n) > t) = P\left(\overline{A}_1 + \cdots + \overline{A}_n\right), . \tag{21}$$

so we have to introduce further notations:

$$\overline{F}_{i_1,\ldots,i_k}(t) = P\left(X_{i_1} > t,\ldots,X_{i_k} > t\right), 1 \le i_1 < \cdots < i_k \le n, \tag{22}$$

$$\overline{S}_k(t) = \sum_{1 \le i_1 < \cdots < i_k \le n} \overline{F}_{i_1,\ldots,i_k}(t), \; k = 1,\ldots,n. \tag{23}$$

Now for $h = 0$ we get the lower bound:

$$
\begin{aligned}
G(t) \quad \geq \quad & 1-\overline{S}_1(t) \\
+ \quad & \sum_{(i_1,i_2)\in_0 \varepsilon_2} \overline{F}_{i_1,i_2}(t) \\
- \quad & \sum_{(i_1,i_2,i_3)\in_0 \varepsilon_3} \overline{F}_{i_1,i_2,i_3}(t) \\
\vdots \quad & \\
+ \quad & (-1)^{m+1} \sum_{(i_1,\ldots,i_{m+1})\in_0 \varepsilon_{m+1}} \overline{F}_{i_1,\ldots,i_{m+1}}(t),
\end{aligned}
\tag{24}
$$

and for $h=1$ we get the upper bound:

$$
\begin{aligned}
G(t) \quad \leq \quad & 1-\overline{S}_1(t)+\overline{S}_2(t) \\
- \quad & \sum_{(i_1,i_2,i_3)\in_1 \varepsilon_2} \overline{F}_{i_1,i_2,i_3}(t) \\
- \quad & \\
\vdots \quad & \\
+ \quad & (-1)^{m} \sum_{(i_1,\ldots,i_{m+2})\in_1 \varepsilon_{m+1}} \overline{F}_{i_1,\ldots,i_{m+2}}(t).
\end{aligned}
\tag{25}
$$

To get the expected value of the elapsed time until failure of the systems one can apply the general integration formula (7) as it was done before.

Conclusions

In this paper the (h,m)-hypermultitree bounds introduced by J. Bukszár were applied to determine lower and upper bounds on the expexted time to the failure of serial and parallel systems. As these bounds proved to be more efficient than the Boole-Bonferroni inequalities are, they are hoped to be useful tools in investigation of failure systems.

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