

On the Support of the Resultant Vector of d'Alembert's Fictitious Forces for Bars and Plates in Rotation Motion

Mihail Boiangiu

Department of Mechanics, "Politehnica" University of Bucharest
Splaiul Independentei 313, sector 6, cod 060042, Bucharest, Romania
E-mail: mboiangiu@gmail.com

Adina Boiangiu

Technical College "Edmond Nicolau", Bucharest, Romania

Abstract: This paper presents the determination of the central axis of d'Alembert's fictitious forces system for plane bars and plates in uniform rotation motion around an axis in their plane. General cases of bars and plates are studied. A general law for the determination of the support of the resultant vector of d'Alembert's fictitious forces is established. This law is based on the position of the center of mass of the rotation surface and body generated by the bar and the plate in rotation, respectively.

Keywords: plate; bar; rotation; d'Alembert's fictitious forces; support of the resultant vector

1 Introduction

A number of problems on the dynamics of rigid bodies [1] are solved by the application of d'Alembert's principle [2], [3], [4], [5].

In order to solve the problems on plates and bars having a uniform rotation motion by using d'Alembert's principle, it is necessary to know the position of the support of the resultant vector of d'Alembert's fictitious forces. Finding the central axis can sometimes be difficult. In the technical literature [4], [5] the support of the resultant vector of d'Alembert's fictitious forces is obtained by the integration of the elementary moment of the d'Alembert's elementary fictitious force and by stating the condition that the resultant moment be zero.

In this paper the authors propose a general method for the determination of the support of the resultant vector of d'Alembert's fictitious forces for bars and plates having a uniform rotation motion. The law proposed is original and is based on the position of the center of mass of the rotation surface and body generated by bar and plate in rotation, respectively.

2 Bars in Rotation Motion

Let us consider a homogeneous plane curve bar (Figure 1a), articulated in point A. The bar rotates with constant angular velocity ω , around the vertical axis which passes through point A and is contained in the plane of the curve.

The system of d'Alembert's fictitious forces (parallel forces) is equivalent to a resultant force on the central axis ($\vec{M}^{in} = 0$) [5]. In these conditions we want to find the support of the resultant vector of d'Alembert's fictitious forces.

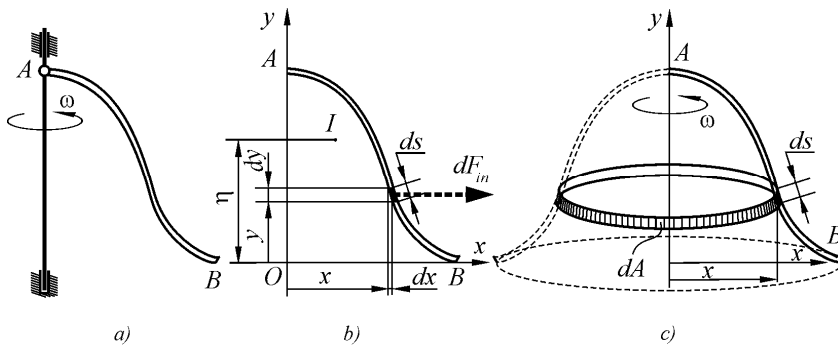


Figure 1

Determination of the support of the resultant vector of d'Alembert's fictitious forces for a bar having a rotation motion: a) bar in rotation motion; b) application of d'Alembert's principle; c) rotation surface generated by the bar in rotation

We relate the curve to a Cartesian reference system (Figure 1b). We isolate an element of infinite little length ds . The elementary d'Alembert's fictitious force dF^{in} is written as follows:

$$dF^{in} = \omega^2 x dm = \omega^2 x \rho ds, \tag{1}$$

where ρ represents the linear density.

If η denotes the y -coordinate of a point I on the support of the resultant vector, then the moment of the d'Alembert's elementary fictitious force with respect to this point will be:

$$dM_I^{in} = dF^{in}(y - \eta) = \omega^2 \rho x(y - \eta) ds. \quad (2)$$

Finally, the resultant moment of d'Alembert's fictitious forces is:

$$M_I^{in} = \int_{(D)} dM_I^{in} = \int_{(D)} \omega^2 \rho x(y - \eta) ds = \rho \omega^2 \int_{(D)} xy ds - \eta \rho \omega^2 \int_{(D)} x ds. \quad (3)$$

As point I is situated on the support of the resultant vector of d'Alembert's fictitious forces, the resultant moment of d'Alembert's fictitious forces is zero. Taking this condition into account, it follows that:

$$\eta = \frac{\rho \omega^2 \int_{(D)} xy ds}{\rho \omega^2 \int_{(D)} x ds}. \quad (4)$$

If we amplify the expression of η by 2π , we obtain the product $2\pi x ds$, which is equal to the elementary area dA generated by the rotation of the element of infinite little length ds (Figure 1c):

$$\eta = \frac{\rho \omega^2 \int_{(D)} xy ds}{\rho \omega^2 \int_{(D)} x ds} = \frac{\int_{(D)} xy ds}{\int_{(D)} x ds} = \frac{\int_{(D)} y 2\pi x ds}{\int_{(D)} 2\pi x ds} = \frac{\int_{(D)} y dA}{\int_{(D)} dA} = y_C. \quad (5)$$

It follows that, for a plane bar in rotation motion, the support of the resultant vector of d'Alembert's fictitious forces passes through the center of mass of the rotation surface generated by the curve in rotation.

Let us consider the example of a homogeneous bar ABO (Figure 2a), consisting of two parts: a circle arch AB of radius R and a straight part $OB = R$. The linear density of the material is ρ (kg/m). The circle arch is articulated in A. The straight part OB is supported in point O. The plane which contains the bar rotates with constant angular velocity ω around the vertical axis that passes through points O and A. We want to find the angular velocity ω so that the bar will not touch the rotation axis in O.

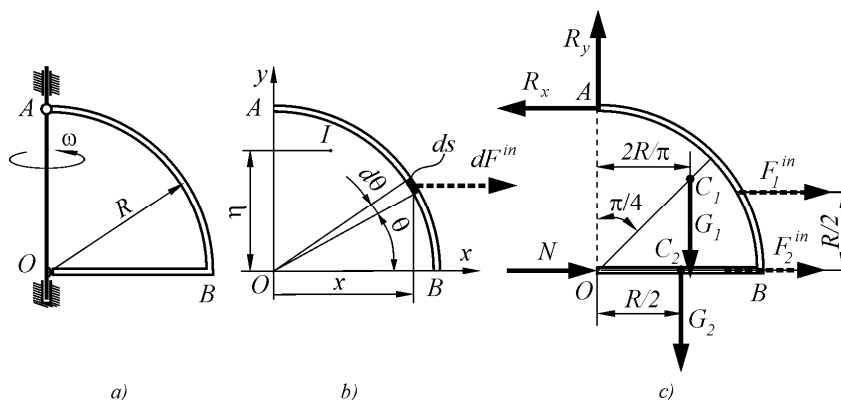


Figure 2

Application for a bar in rotation motion: a) bar in rotation motion; b) determination of the support of the resultant vector of d'Alembert's fictitious forces by the "classic" method; c) application of relation(8)

The support of the resultant vector of d'Alembert's fictitious forces for the circle arch AB can be determined by using relation (5), or in the "classic" way.

First we will use the "classic" method [4]. Let us consider an element of the bar of length ds which corresponds to an angle $d\theta$ (Figure 2b). For this element the d'Alembert's elementary fictitious force is:

$$dF^{in} = \omega^2 x dm = \omega^2 x \rho ds = \omega^2 \cdot R \cos \theta \cdot \rho \cdot R d\theta = \rho \omega^2 R^2 \cos \theta d\theta . \quad (6)$$

We consider that the support of the resultant vector of d'Alembert's fictitious forces crosses the rotation axis at distance η from the center of the circle arch AB. The moment of the d'Alembert's elementary fictitious force with respect to a point I , on the support of the resultant vector, is:

$$dM_I^{in} = dF^{in} (\eta - R \sin \theta) = \rho \omega^2 R^2 \cos \theta (\eta - R \sin \theta) d\theta . \quad (7)$$

The resultant moment will be:

$$\begin{aligned} M_I^{in} &= \int_{(D)} dM_I^{in} = \int_0^{\pi/2} \rho \omega^2 R^2 \cos \theta (\eta - R \sin \theta) d\theta = \rho \omega^2 R^2 \eta \int_0^{\pi/2} \cos \theta d\theta - \\ &- \rho \omega^2 R^3 \int_0^{\pi/2} \sin \theta \cos \theta d\theta = \rho \omega^2 R^2 \eta \int_0^{\pi/2} \cos \theta d\theta - \frac{1}{2} \rho \omega^2 R^3 \int_0^{\pi/2} \sin 2\theta d\theta . \end{aligned} \quad (8)$$

With the change of variable $u = 2\theta$, from relation (8) we obtain:

$$M_I^{in} = \rho \omega^2 R^2 \eta + \frac{1}{4} \rho \omega^2 R^3 \int_0^{\pi} (-\sin u) du = \rho \omega^2 R^2 \eta - \frac{1}{2} \rho \omega^2 R^3 . \quad (9)$$

The resultant moment M_I^{in} is zero on the support of the resultant vector of d'Alembert's fictitious forces. From relation (9) it results that $\eta = R/2$.

The same result can be quickly obtained by applying relation (5). By rotation, the curve bar AB generates a hemisphere surface. This is a special case of a spherical zone. The center of mass is on the rotation axis at the distance $R/2$ from the center of the sphere. The support of the resultant vector of d'Alembert's fictitious forces crosses the rotation axis in this point.

The resultant vector of d'Alembert's fictitious forces is:

$$F_I^{in} = m_{OA} a_{C_1}^v = \rho \frac{\pi R}{2} \frac{R \sin \frac{\pi}{4}}{\frac{\pi}{4}} \sin \frac{\pi}{4} \omega^2 = \rho R^2 \omega^2. \quad (10)$$

The force of gravity is:

$$G_I = \rho g l_{OA} = \frac{\rho \pi R g}{2}. \quad (11)$$

For the straight bar OB the support of the resultant vector of d'Alembert's fictitious forces is in the direction of the line OB and crosses the rotation axis in point O.

The resultant vector of d'Alembert's fictitious forces for the straight bar OB is:

$$F_2^{in} = m_{AB} a_{C_2}^v = \frac{\rho R^2 \omega^2}{2}. \quad (12)$$

The force of gravity corresponding to the straight bar OB is:

$$G_2 = \rho g l_{AB} = \rho R g. \quad (13)$$

Now we will apply d'Alembert's principle. We isolate the bar ABO and we obtain the system of forces in Figure 2c. We write the moment's equation with respect to point A:

$$N \cdot R + \rho R^2 \omega^2 \cdot \frac{R}{2} + \frac{\rho R^2 \omega^2}{2} \cdot R - \frac{\rho \pi R g}{2} \cdot \frac{2R}{\pi} - \rho R g \cdot \frac{R}{2} = 0. \quad (14)$$

We obtain the expression for the reaction in point O:

$$N = \rho R \left(\frac{3}{2} g - R \omega^2 \right). \quad (15)$$

From the condition $N \geq 0$ we obtain the value for the angular velocity ω :

$$\omega \geq \sqrt{\frac{3g}{2R}}. \quad (16)$$

3 Plates in Rotation Motion

Let us consider a homogeneous plane plate (Figure 3a). This plate rotates with a constant angular velocity ω , around a vertical axis which is identical to a straight leg. The axis is contained in the plane of the plate.

The system of d'Alembert's fictitious forces (parallel forces) is reduced to a resultant vector on the central axis ($\vec{M}^{in} = 0$) [5]. In these conditions we want to find the support of the resultant vector of d'Alembert's fictitious forces.

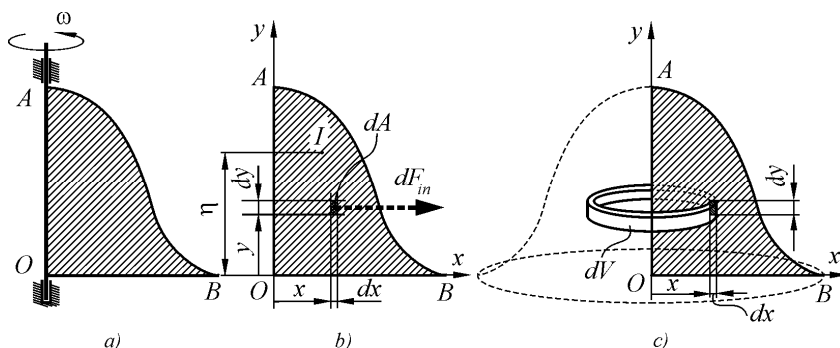


Figure 3

Determination of the support of the resultant vector of d'Alembert's fictitious forces for a plate in rotation motion: a) plate in rotation motion; b) application of d'Alembert's principle; c) rotation body generated by plate in rotation

We relate the plate to a Cartesian reference system (Figure 3b). We isolate an element of infinite little area $dA=dx dy$. The elementary d'Alembert's fictitious force dF^{in} is written:

$$dF^{in} = \omega^2 x dm = \omega^2 x \rho dA = \omega^2 x \rho dx dy , \tag{17}$$

where ρ represents the surface density.

If η denotes the y -coordinate of a point I on the support of the resultant vector, then the moment of the d'Alembert's elementary fictitious force with respect to this point will be:

$$dM_I^{in} = dF^{in} (y - \eta) = \omega^2 \rho x (y - \eta) dx dy . \tag{18}$$

Finally, the resultant moment of d'Alembert's fictitious forces is:

$$M_I^{in} = \int_{(D)} dM_I^{in} = \int_{(D)} \omega^2 \rho x (y - \eta) dx dy = \rho \omega^2 \int_{(D)} xy dx dy - \eta \rho \omega^2 \int_{(D)} x dx dy . \tag{19}$$

As point I is situated on the support of the resultant vector of d'Alembert's fictitious forces, the resultant moment of these forces is zero. Taking this condition into account it follows that:

$$\eta = \frac{\rho\omega^2 \int_{(D)} xy \, dx dy}{\rho\omega^2 \int_{(D)} x \, dx dy} \quad (20)$$

If we amplify the expression of η by 2π , we obtain the product $2\pi x \, dx dy$, which is equal to the elementary volume dV generated in rotation by the element of infinite little area dA (Figure 3c):

$$\eta = \frac{\rho\omega^2 \int_{(D)} xy \, dx dy}{\rho\omega^2 \int_{(D)} x \, dx dy} = \frac{\int_{(D)} xy \, dx dy}{\int_{(D)} x \, dx dy} = \frac{\int_{(D)} y 2\pi x \, dx dy}{\int_{(D)} 2\pi x \, dx dy} = \frac{\int_{(D)} y \, dV}{\int_{(D)} dV} = y_C \quad (21)$$

It results that, for a plane plate in rotation motion, the support of the resultant vector of d'Alembert's fictitious forces passes through the center of mass of the rotation body generated by the plate in rotation.

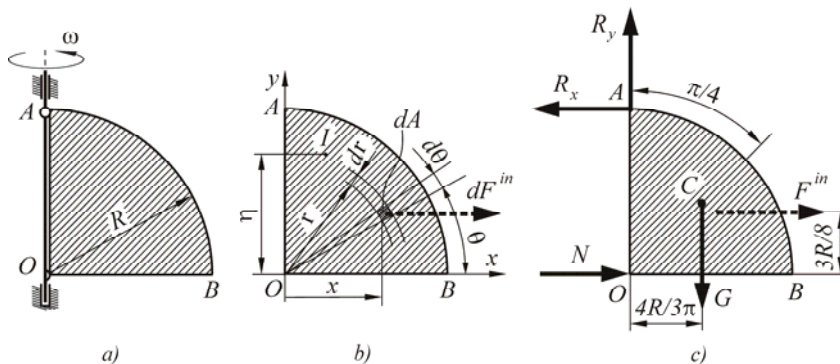


Figure 4

Application for a plate in rotation motion: a) plate in rotation motion; b) determination of the support of the resultant vector of d'Alembert's fictitious forces by "classic" method; c) application of relation(21)

Let us consider the example of a homogeneous plane plate OAB, a quarter of disc (Figure 4a) of radius R . The surface density of the material is ρ (kg/m^2). The plate rotates with constant angular velocity ω around the vertical axis that passes through points O and A. We want to find the angular velocity ω so that the plate will not touch the rotation axis in O.

The resultant vector support of d'Alembert's fictitious forces can be found using the "classic" method or relation (21).

First we will use the “classic” method [4]. Let us consider an element of the plate of area dA which corresponds to an angle $d\theta$ and a radius r (Figure 4b). For this element the d'Alembert's elementary fictitious force is:

$$dF^{in} = \omega^2 x dm = \omega^2 x \rho dA = \omega^2 \cdot r \cos \theta \cdot \rho \cdot r d\theta dr = \rho \omega^2 r^2 \cos \theta dr d\theta. \quad (22)$$

We consider that the support of the resultant vector of d'Alembert's fictitious forces crosses the rotation axis at the distance η from the center O. The moment of the d'Alembert's elementary fictitious force with respect to a point I, on the support of the resultant vector, is:

$$dM_I^{in} = dF^{in} (\eta - r \sin \theta) = \rho \omega^2 r^2 \cos \theta (\eta - r \sin \theta) dr d\theta. \quad (23)$$

The resultant moment will be:

$$M_I^{in} = \iint_{(D)} dM_I^{in} = \int_0^R \int_0^{\pi/2} \rho \omega^2 r^2 \cos \theta (\eta - r \sin \theta) dr d\theta = \rho \omega^2 \eta \int_0^R \int_0^{\pi/2} r^2 \cos \theta dr d\theta - \rho \omega^2 \int_0^R \int_0^{\pi/2} r^3 \sin \theta \cos \theta dr d\theta = \rho \omega^2 \left(\eta \int_0^R \int_0^{\pi/2} r^2 \cos \theta dr d\theta - \frac{1}{2} \int_0^R \int_0^{\pi/2} r^3 \sin 2\theta dr d\theta \right). \quad (24)$$

With the change of variable $u = 2\theta$, from relation (24) we obtain:

$$M_I^{in} = \rho \omega^2 \frac{r^3}{3} \eta + \frac{1}{4} \rho \omega^2 \int_0^R \int_0^{\pi} r^3 (-\sin u) dr du = \rho \omega^2 \frac{r^3}{3} \eta - \frac{1}{8} \rho \omega^2 R^4. \quad (25)$$

The resultant moment M_I^{in} is zero on the support of the resultant vector of d'Alembert's fictitious forces. From relation (25) it follows that $\eta = \frac{3}{8} R$.

We obtain the same result by applying relation (21). By rotation, the plate generates a hemisphere. The centre of mass is on the rotation axis at the distance $\frac{3}{8} R$ from the center of the sphere. In this point the support of the resultant vector of d'Alembert's fictitious forces crosses the rotation axis.

The resultant vector of d'Alembert's fictitious forces is:

$$F^{in} = m a_C^v = \rho \frac{\pi R^2}{4} \cdot \frac{2}{3} \frac{R \sin \frac{\pi}{4}}{\frac{\pi}{4}} \sin \frac{\pi}{4} \omega^2 = \frac{1}{3} \rho R^3 \omega^2. \quad (26)$$

The force of gravity is:

$$G = \rho A g = \frac{1}{4} \rho \pi R^2 g. \quad (27)$$

Now we will apply d'Alembert's principle. We isolate the plate OAB and we obtain the system of forces in Figure 4c. We write the moment's equation with respect to point A:

$$N \cdot R + \frac{1}{3} \rho R^3 \omega^2 \cdot \frac{5}{8} R - \frac{1}{4} \rho \pi R^2 g \cdot \frac{4R}{3\pi} = 0. \quad (28)$$

We obtain the expression for the reaction in point O:

$$N = \rho R^2 \left(\frac{1}{3} g - \frac{5}{24} R \omega^2 \right). \quad (29)$$

From the condition $N \leq 0$ we obtain the value for the angular velocity ω :

$$\omega \geq \sqrt{\frac{8g}{5R}}. \quad (30)$$

Conclusions

In this paper the authors have proposed a rule for the determination of the support of the resultant vector of d'Alembert's fictitious forces.

To sum up, for a plane bar having a rotation motion, the support of the resultant vector of d'Alembert's fictitious forces passes through the center of mass of the rotation surface generated by the curve in rotation. Also, for a plane plate in rotation motion, the support of the resultant vector of d'Alembert's fictitious forces passes through the center of mass of the rotation body generated by the plate in rotation.

If we consider the fact that the positions of the center of mass for a large number of bodies can be found in the technical literature, the method proposed here is accessible because it replaces the integral calculus.

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